XXV. *Of Cubic Equations and Infinite Series.* *By* Charles Hutton, *LL.D. F. R. S.*

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THE following pages are not to be underſtood as intended to contain a complete treatiſe on cubic equations, with all the methods of ſolution that have been delivered by other writers; but they are chiefly employed on the improvements of ſome properties that were before but partially known, with the diſcovery of ſeveral others which to me appear to be new and of no ſmall importance: for I have only ſlightly mentioned ſuch of the generally known properties as were neceſſary to the introduction or inveſtigation of the many curious conſequences herein deduced from them.

Art. 1. Every equation, whoſe terms are expreſſed in ſimple integral powers, has as many roots as there are dimenſions in the higheſt power. And when all the terms are brought to one ſide of the equation, and the

 coefficient

coefficient of the firſt term or higheſt power is + 1, then the coefficient of the ſecond term is equal to the ſum of all the roots with contrary ſigns; the coefficient of the third term is equal to the ſum of all the products made by multiplying every two of the roots together; the coefficient of the fourth term, to the ſum of all the products ariſing from the multiplication of every three of the roots together; &c. and the laſt term, equal to the continual product of all the roots; the ſigns of all of them being ſuppoſed to be changed into the contrary ſigns before theſe multiplications are made. All this is evident from the generation of equations. And from theſe properties of the coefficients the following deductions are eaſily made.

2. If the ſigns of all the roots of an equation be changed, and another equation be generated from the ſame roots with the ſigns ſo changed; the terms of this laſt equation will have the ſame coefficients as the former, only the ſigns of all the even terms will be changed, but not thoſe of the odd terms: for the coefficients of the ſecond, fourth, and the other even terms, are made up of products conſiſting each of an odd number of factors; while thoſe of the third, fifth, and other odd terms are compoſed of products having an even number of factors: and the change of the ſigns of all the factors

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produces a change in the ſign of the continual product of an odd number of factors, but no change in the ſign of that of an even number of factors. Wherefore, changing the ſigns of all the even terms, namely, the ſecond, fourth, &c. produces no alteration in the roots, but only in their ſigns, the poſitive roots being changed into negative, and the negative into poſitive. But by changing any or all the ſigns of the odd terms, the equation will no longer have the ſame roots as before, but will have new roots of very different magnitudes from thoſe of the former, unleſs the ſign of the firſt term or higheſt power is changed alſo; but this term is always to be ſuppoſed to remain poſitive.

3. It alſo follows, that when any term is wanting in an equation, or the coefficient of any term equal to o, the ſum of the negative products in the coefficient of that term is equal to the ſum of the poſitive products in the ſame. And if it be the ſecond term which is wanting, then the equation has both negative and poſitive roots, and the ſum of the negative roots is equal to the ſum of the poſitive ones. But if it be the laſt term which is wanting, then one of the roots of the equation is equal to nothing. And hence ariſes a method of tranſforming any equation into another which ſhall want the ſecond term: and to this latter ſtate it will be proper to

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transform every cubic equation before we attempt the resolution of it.

4. Let therefore $x^3 + px = q$ be such a cubic equation wanting the second term, where p and q represent any numbers, positive or negative.

5. Now from the premises it follows, that this equation has three roots; that some are positive, and others negative; that two of them are of one affection, and are together equal to the third of a contrary affection, namely, either two negative roots, which are together equal to the other positive, or two positive roots equal to the third negative.

6. But the signs of the three roots are easily known from the sign of the quantity q; the sign of the greatest root being the same with the sign of q when this quantity is on the right-hand side of the equation, and the other two roots of the contrary sign. For when q is on the same side of the equation with the other terms, it has been observed, that it is always equal to the continual product of all the roots with their signs changed; consequently q is equal to the product of all the roots under their own signs, when that quantity is on the other or right-hand side of the equation: but the product of the two less roots is always positive, because they are of the same affection, either both + or both −; and therefore

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this product, drawn into the third or greatest root, will generate another product equal to q, and of the same affection with this root.

7. But the roots of equations of the above form are not only positive, negative, or nothing, but sometimes also imaginary. We have found that the greatest root is positive when q is positive, and negative when it is negative; as also that one root is = to o when q is = o, and in this case the other two roots must be equal to each other, with contrary signs. But to discover the cases in which the equation has imaginary roots, as well as many other properties of the equation, it will be proper to consider the generation of it as follows.

8. The roots of equations becoming imaginary in pairs, the number of imaginary roots is always even; and therefore the cubic equation has either two imaginary roots, or none at all; and consequently it has at least one real root. Let that root be represented by r, which may be either positive or negative, and may be any one of the real roots, when none of them are imaginary: then since any one of the roots is equal to the sum of the other two with their signs changed, the other two roots may be represented by $-\frac{1}{2}r \pm$ some other quantity, since the sum of these two, with the signs changed, is $= r$. Now this supplemental quantity, which

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is to be connected with $-\frac{1}{2}r$ by the signs + and − to compose the other two roots, will be a real quantity when those roots are real, but an imaginary one when they are imaginary, since the other part $(-\frac{1}{2}r)$ of those two roots is real by the hypothesis. Let this supplemental quantity be represented by e when it is real, or $e\sqrt{-1}$ or $\sqrt{-e^2}$ when it is imaginary: we shall use the quantity e in what follows for the real roots; and it is evident, that by changing e for $e\sqrt{-1}$, or e^2 for $-e^2$, that is, by barely changing the sign of e^2 wherever it is found, the expressions will become adapted to the imaginary roots. Hence then the three roots are represented by r, and $-\frac{1}{2}r+e$, and $-\frac{1}{2}r-e$; and consequently the three equations, from whose continual multiplication by one another the cubic equation is to be generated, will be $x-r=0$, and $x+\frac{1}{2}r-e=0$, and $x+\frac{1}{2}r+e=0$.

9. Let now these three equations be multiplied together, and there will be produced this general cubic equation wanting the second term, namely, $x^3 \begin{smallmatrix}-\frac{3}{4}r^2\\ -e^2\end{smallmatrix} x - r\,.\,\overline{\tfrac{1}{4}r^2-e^2}=0$, or $x^3 \begin{smallmatrix}-\frac{3}{4}r^2\\ -e^2\end{smallmatrix} x = r\,.\,\overline{\tfrac{1}{4}r^2-e^2}$, having three real roots; and if the sign of e^2 be changed from − to +, it will then represent all the cases which have only one real and two imaginary roots: and from the bare inspection of this equation the following properties are easily drawn.

10. First,

10. Firſt, we hence find, that when the equation has three real roots, the ſign of the ſecond term is always $-$; for the coefficient of that term, or p is $= -\frac{3}{4}r^2 - e^2$, which is always negative when r and e are real quantities. And conſequently when p is poſitive, the equation has two imaginary roots, ſince $-p$ includes all the caſes of three real roots. But it does not therefore follow, that when p is negative, the three roots are always real; and indeed there are imaginary roots not only whenever p is poſitive, but ſometimes alſo when p is negative: for ſince p is $= -\frac{3}{4}r^2 - e^2$ in all the caſes of three real roots, it will be $p = -\frac{3}{4}r^2 + e^2$ for all the caſes of two imaginary roots; and it is evident, that p will be either poſitive or negative, according as e^2 is greater or leſs than $\frac{3}{4}r^2$.

11. But to find the caſes of $-p$ when the roots are all real, and when not, will require ſome farther conſideration; and in order to that it muſt be obſerved, that e^2 ought to be poſitive and leſs than $\frac{3}{4}r^2$; but the limit between the caſes of real and imaginary roots is when $e^2 = 0$, or $e = 0$; and then p becomes $= -\frac{3}{4}r^2$, and $q = \frac{1}{4}r^3$; conſequently then $\overline{\tfrac{1}{3}p}|^3 = \overline{\tfrac{1}{4}r^2}|^3 = \frac{1}{64}r^6$, which is $= \overline{\tfrac{1}{2}q}|^2 = \overline{\tfrac{1}{8}r^3}|^2 = \frac{1}{64}r^6$, that is, when e is $= 0$, then $\overline{\tfrac{1}{3}p}|^3$ is $= \overline{\tfrac{1}{2}q}|^2$, and conquently when $\overline{\tfrac{1}{3}p}|^3$ is leſs than $\overline{\tfrac{1}{2}q}|^2$, the equation has two imaginary roots, otherwiſe none, the ſign of p being $-$.

Thus

Thus then we eafily perceive in all cafes the nature of the roots as to real and imaginary; namely, partly from the fign of p, and partly from the relation of p to q: for the equation has always two imaginary roots when p is pofitive; it has alfo two imaginary roots when p is negative, and $\overline{\tfrac{1}{3}p}|^3$ lefs than $\overline{\tfrac{1}{2}q}|^2$; in the other cafe the roots are all real, namely, when p is negative and $\overline{\tfrac{1}{3}p}|^3$ either equal to or greater than $\overline{\tfrac{1}{2}q}|^2$.

12. Moreover, when p is $= 0$, the equation has two imaginary roots; for this cannot happen but by $-e^2$ becoming $+e^2$, in the value of p, and $=$ to $\tfrac{3}{4}r^2$; and then $p = -\tfrac{3}{4}r^2 + e^2 = -\tfrac{3}{4}r^2 + \tfrac{3}{4}r^2 = 0$, and $q = r \,.\, \overline{\tfrac{1}{4}r^2 + e^2} = r \,.\, \overline{\tfrac{1}{4}r^2 + \tfrac{3}{4}r^2} = r \,.\, r^2 = r^3$, and confequently the above becomes barely $x^3 = r^3$, which therefore, befides one real general equation root $x = r$, has alfo two imaginary roots.

13. Hence alfo it again appears, that the greateft root is always of the fame affection, in refpect of pofitive and negative, with q on the right-hand fide of the equation, they being either both pofitive or both negative together; and the other two roots of the contrary fign. For if r be the greateft root, then is $\tfrac{1}{2}r$ greater than e, and $\tfrac{1}{4}r^2$ greater than e^2, and $\tfrac{1}{4}r^2 - e^2$ always pofitive, and confequently the product $r \,.\, \overline{\tfrac{1}{4}r^2 - e^2}$, or q, will have the fame fign with r. But if r be one of the lefs roots, the contrary

trary of this will happen; for then $\frac{1}{2}r$ is lefs than e, and confequently $\frac{1}{4}r^2$ lefs than e^2, and fo $\overline{\frac{1}{4}r^2 - e^2}$ a negative quantity, and therefore the product $r \cdot \overline{\frac{1}{4}r^2 - e^2}$, or q, will have the fign contrary to that of r; that is, q and the lefs roots have different figns, and confequently q and the greateft root the fame fign, fince the fign of the greateft root is always contrary to that of the other two roots.

14. Moreover, when q or $r \cdot \overline{\frac{1}{4}r^2 - e^2}$ is pofitive, then r denotes the greateft root; for then $\frac{1}{4}r^2$ is greater than e^2, or $\frac{1}{2}r$ greater than e, and r greater than either $-\frac{1}{2}r + e$ or $-\frac{1}{2}r - e$. But when q or $r \cdot \overline{\frac{1}{4}r^2 - e^2}$ is negative, then r reprefents one of the other two roots in the equation; fince then e is greater than $\frac{1}{2}r$, and $-\frac{1}{2}r - e$ greater than r. Laftly, when q is between the pofitive and negative ftates, or $q = 0$, then r ought to be neither the greateft nor one of the lefs roots, if I may fo fpeak, that is, two of the roots are equal, and the third root $= 0$, fince then $\frac{1}{4}r^2$ muft be $= e^2$, or $\frac{1}{2}r = e$.

15. Hence it appears, that the fign of p determines the nature of the roots as to real and imaginary, and the fign of q determines the affection of the roots as to pofitive and negative. Let us illuftrate thefe rules by a few examples.

16. The equation $x^3 - 9x = 10$ has all its three roots real, becaufe $p = -9$ is negative, and $\overline{\frac{1}{3}p}|^3 = 3^3 = 27$ is

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greater than $\overline{\tfrac{1}{2}q}|^2$, which is $= 5^2 = 25$; and the greateſt of the roots is poſitive, becauſe $q = 10$ is poſitive; and the two leſs roots negative.

17. The equation $x^3 - 9x = -10$ has the ſame three real roots as the former, but with the contrary ſigns, the ſign of the greateſt root being now negative, becauſe $q = -10$ is negative.

18. But the equation $x^3 + 9x = \pm 10$ has only one real root and two imaginary roots, becauſe $p = 9$ is poſitive; and the ſign of the real root is + or – according as the ſign of q or 10 is + or –.

19. The equation $x^3 + 6x = \pm 10$ has alſo two imaginary roots, and one real root, which is + or – as it is +10 or –10, for the ſame reaſon as before.

20. The equation $x^3 - 6x = \pm 10$ has alſo two imaginary roots, becauſe $\overline{\tfrac{1}{3}p}|^3 = 2^3 = 8$ is leſs than $\overline{\tfrac{1}{2}q}|^2 = 5^2 = 25$.

21. But the equation $x^3 - 12x = \pm 16$ has all its roots real, becauſe $\overline{\tfrac{1}{3}p}|^3 = 4^3 = 64$ is $= \overline{\tfrac{1}{2}q}|^2 = 8^2 = 64$.

22. And the equation $x^3 + 12x = \pm 16$ has only one real root, becauſe $p = +12$ is poſitive.

23. Let us now conſider the other properties and relations of the roots ariſing from certain aſſumed relations between e and r, and from conſidering e either as real, imaginary, or nothing, that is e^2 as poſitive, negative, or nothing.

24. When

24. When e is a real quantity, the general equation is $x^3 {}^{-\frac{3}{4}r^2}_{-e^2} x = r \cdot \overline{\frac{1}{4}r^2 - e^2}$, and all the roots are real.

25. When e is imaginary, the general equation is $x^3 {}^{-\frac{3}{4}r^2}_{+e^2} x = r \cdot \overline{\frac{1}{4}r^2 + e^2}$, and two of the roots are imaginary.

26. When e is between thefe two ftates, or $= 0$, the equation becomes $x^3 - \frac{3}{4}r^2 x = \frac{1}{4}r^3$, and the root $r = \sqrt{\frac{4}{3}p}$ $= \sqrt[3]{4q} = \frac{3q}{p}$; for in this cafe $p = \frac{3}{4}r^2$, and $q = \frac{1}{4}r^3$. Alfo the other two roots $-\frac{1}{2}r \pm e$ are each $= -\frac{1}{2}r$.

27. Affume now any general relation between the root r and the fupplemental part e of the other two roots, as fuppofe $r^2 : e^2 :: 4 : n$, or $e^2 = \frac{n}{4}r^2$, or $e = \frac{1}{2}r\sqrt{n}$, where n reprefents either nothing or any quantity whether pofitive or negative, that is, pofitive when e and all the three roots are real, or negative when e and two of the roots are imaginary. Subftitute now $\frac{n}{4}r^2$ inftead of e^2 in the general equation $x^3 {}^{-\frac{3}{4}r^2}_{-e^2} x = r \cdot \overline{\frac{1}{4}r^2 - e^2}$, and that equation will become $x^3 - \frac{3+n}{4}r^2 x = \frac{1-n}{4}r^3$. Here then $p = \frac{3+n}{4}r^2$, and $q = \frac{1-n}{4}r^3$, and confequently the root $r =$ $\sqrt{\frac{4p}{3+n}} = \sqrt[3]{\frac{4q}{1-n}} = \frac{n+3}{n-1} \cdot \frac{q}{p}$ expreffed in three different ways. The other roots, the general values of which are $-\frac{1}{2}r \pm e$, become $-\frac{1}{2}r \pm \sqrt{\frac{n}{4}r^2} = -\frac{1}{2}r \pm \frac{1}{2}r\sqrt{n} = -\frac{1}{2}r \times \overline{1 \pm \sqrt{n}}$.

 28. Hence

28. Hence then in an eafy and general manner we can reprefent any form or cafe of the general equation, with all the circumftances of the roots, by only taking, in thefe laft formulæ, any particular number for n, either pofitive or negative, integral or fractional, &c. As if $n = 1$; then the equation becomes $x^3 - r^2 x = \frac{0}{2} r^3$, or $= 0$, the value of $e = \frac{1}{2} r$, the root $r = \sqrt{p} = \sqrt[3]{\frac{4q}{0}} = \frac{4q}{0p}$, and the other two roots $= -\frac{1}{2} r . \overline{1 \pm \sqrt{1}} = -\frac{1}{2} r . 2$ and $-\frac{1}{2} r . 0 = -r$ and 0.

29. If $n = -1$, the equation will be $x^3 - \frac{1}{2} r^2 x = \frac{1}{2} r^3$, the value of $e = \frac{1}{2} r \sqrt{-1}$, the root $r = \frac{q}{p} = \sqrt{2p} = \sqrt[3]{2q}$, and the other two roots $= -\frac{1}{2} r . \overline{1 \pm \sqrt{1}}$ imaginary.

30. And thus, by taking feveral different values of n, pofitive and negative, the various correfponding circumftances and relations of the equation and roots will be exhibited as in the following table.

Forms

Forms or cases.	Values of *n*.	Values of *e*.	Forms of the equation.	Values of the root *r*, viz. *r* =	Values of the two other roots, viz. $-\frac{1}{2}r \times$
	$+n$	$\frac{1}{2}r\sqrt{+n}$	$x^3 - \frac{n+3}{4}r^2x = -\frac{n-1}{4}r^3$	$\frac{n+3 \cdot q}{n-1\,p} = \sqrt{\frac{4p}{n+3}} = \sqrt[3]{\frac{4q}{n-1}}$	$1 \pm \sqrt{+n}$
1	$+12$	$\frac{1}{2}r\sqrt{+12}$	$x^3 - \frac{15}{4}r^2x = -\frac{11}{4}r^3$	$\frac{15q}{11p} = \sqrt{\frac{4p}{15}} = \sqrt[3]{\frac{4q}{11}}$	$1 \pm \sqrt{+12}$
2	$+11$	$\frac{1}{2}r\sqrt{+11}$	$x^3 - \frac{14}{4}r^2x = -\frac{10}{4}r^3$	$\frac{14q}{10p} = \sqrt{\frac{4p}{14}} = \sqrt[3]{\frac{4q}{10}}$	$1 \pm \sqrt{+11}$
3	$+10$	$\frac{1}{2}r\sqrt{+10}$	$x^3 - \frac{13}{4}r^2x = -\frac{9}{4}r^3$	$\frac{13q}{9p} = \sqrt{\frac{4p}{13}} = \sqrt[3]{\frac{4q}{9}}$	$1 \pm \sqrt{+10}$
4	$+9$	$\frac{1}{2}r\sqrt{+9}$	$x^3 - \frac{12}{4}r^2x = -\frac{8}{4}r^3$	$\frac{12q}{8p} = \sqrt{\frac{4p}{12}} = \sqrt[3]{\frac{4q}{8}}$	$1 \pm \sqrt{+9}$
5	$+8$	$\frac{1}{2}r\sqrt{+8}$	$x^3 - \frac{11}{4}r^2x = -\frac{7}{4}r^3$	$\frac{11q}{7p} = \sqrt{\frac{4p}{11}} = \sqrt[3]{\frac{4q}{7}}$	$1 \pm \sqrt{+8}$
6	$+7$	$\frac{1}{2}r\sqrt{+7}$	$x^3 - \frac{10}{4}r^2x = -\frac{6}{4}r^3$	$\frac{10q}{6p} = \sqrt{\frac{4p}{10}} = \sqrt[3]{\frac{4q}{6}}$	$1 \pm \sqrt{+7}$
7	$+6$	$\frac{1}{2}r\sqrt{+6}$	$x^3 - \frac{9}{4}r^2x = -\frac{5}{4}r^3$	$\frac{9q}{5p} = \sqrt{\frac{4p}{9}} = \sqrt[3]{\frac{4q}{5}}$	$1 \pm \sqrt{+6}$
8	$+5$	$\frac{1}{2}r\sqrt{+5}$	$x^3 - \frac{8}{4}r^2x = -\frac{4}{4}r^3$	$\frac{8q}{4p} = \sqrt{\frac{4p}{8}} = \sqrt[3]{\frac{4q}{4}}$	$1 \pm \sqrt{+5}$
9	$+4$	$\frac{1}{2}r\sqrt{+4}$	$x^3 - \frac{7}{4}r^2x = -\frac{3}{4}r^3$	$\frac{7q}{3p} = \sqrt{\frac{4p}{7}} = \sqrt[3]{\frac{4q}{3}}$	$1 \pm \sqrt{+4}$
10	$+3$	$\frac{1}{2}r\sqrt{+3}$	$x^3 - \frac{6}{4}r^2x = -\frac{2}{4}r^3$	$\frac{6q}{2p} = \sqrt{\frac{4p}{6}} = \sqrt[3]{\frac{4q}{2}}$	$1 \pm \sqrt{+3}$
11	$+2$	$\frac{1}{2}r\sqrt{+2}$	$x^3 - \frac{5}{4}r^2x = -\frac{1}{4}r^3$	$\frac{5q}{1p} = \sqrt{\frac{4p}{5}} = \sqrt[3]{\frac{4q}{1}}$	$1 \pm \sqrt{+2}$
12	$+1$	$\frac{1}{2}r\sqrt{+1}$	$x^3 - \frac{4}{4}r^2x = \mp \frac{0}{4}r^3$	$\frac{4q}{0p} = \sqrt{\frac{4p}{4}} = \sqrt[3]{\frac{4q}{0}}$	$1 \pm \sqrt{+1}$
13	± 0	$\frac{1}{2}r\sqrt{\pm 0}$	$x^3 - \frac{3}{4}r^2x = +\frac{1}{4}r^3$	$\frac{3q}{1p} = \sqrt{\frac{4p}{3}} = \sqrt[3]{\frac{4q}{1}}$	$1 \pm \sqrt{\pm 0}$
14	-1	$\frac{1}{2}r\sqrt{-1}$	$x^3 - \frac{2}{4}r^2x = +\frac{2}{4}r^3$	$\frac{2q}{2p} = \sqrt{\frac{4p}{2}} = \sqrt[3]{\frac{4q}{2}}$	$1 \pm \sqrt{-1}$
15	-2	$\frac{1}{2}r\sqrt{-2}$	$x^3 - \frac{1}{4}r^2x = +\frac{3}{4}r^3$	$\frac{1q}{3p} = \sqrt{\frac{4p}{1}} = \sqrt[3]{\frac{4q}{3}}$	$1 \pm \sqrt{-2}$
16	-3	$\frac{1}{2}r\sqrt{-3}$	$x^3 \mp \frac{0}{4}r^2x = +\frac{4}{4}r^3$	$\frac{0q}{4p} = \sqrt{\frac{4p}{0}} = \sqrt[3]{\frac{4q}{4}}$	$1 \pm \sqrt{-3}$
17	-4	$\frac{1}{2}r\sqrt{-4}$	$x^3 + \frac{1}{4}r^2x = +\frac{5}{4}r^3$	$\frac{1q}{5p} = \sqrt{\frac{4p}{1}} = \sqrt[3]{\frac{4q}{5}}$	$1 \pm \sqrt{-4}$
18	-5	$\frac{1}{2}r\sqrt{-5}$	$x^3 + \frac{2}{4}r^2x = +\frac{6}{4}r^3$	$\frac{2q}{6p} = \sqrt{\frac{4p}{2}} = \sqrt[3]{\frac{4q}{6}}$	$1 \pm \sqrt{-5}$
19	-6	$\frac{1}{2}r\sqrt{-6}$	$x^3 + \frac{3}{4}r^2x = +\frac{7}{4}r^3$	$\frac{3q}{7p} = \sqrt{\frac{4p}{3}} = \sqrt[3]{\frac{4q}{7}}$	$1 \pm \sqrt{-6}$
20	-7	$\frac{1}{2}r\sqrt{-7}$	$x^3 + \frac{4}{4}r^2x = +\frac{8}{4}r^3$	$\frac{4q}{8p} = \sqrt{\frac{4p}{4}} = \sqrt[3]{\frac{4q}{8}}$	$1 \pm \sqrt{-7}$
21	-8	$\frac{1}{2}r\sqrt{-8}$	$x^3 + \frac{5}{4}r^2x = +\frac{9}{4}r^3$	$\frac{5q}{9p} = \sqrt{\frac{4p}{5}} = \sqrt[3]{\frac{4q}{9}}$	$1 \pm \sqrt{-8}$
22	-9	$\frac{1}{2}\sqrt{-9}$	$x^3 + \frac{6}{4}r^2x = +\frac{10}{4}r^3$	$\frac{6q}{10p} = \sqrt{\frac{4p}{6}} = \sqrt[3]{\frac{4q}{10}}$	$1 \pm \sqrt{-9}$
23	-10	$\frac{1}{2}r\sqrt{-10}$	$x^3 + \frac{7}{4}r^2x = +\frac{11}{4}r^3$	$\frac{7q}{11p} = \sqrt{\frac{4p}{7}} = \sqrt[3]{\frac{4q}{11}}$	$1 \pm \sqrt{-1}$
24	-11	$\frac{1}{2}r\sqrt{-1}$	$x^3 + \frac{8}{4}r^2x = +\frac{12}{4}r^3$	$\frac{8q}{12p} = \sqrt{\frac{4p}{8}} = \sqrt[3]{\frac{4q}{12}}$	$1 \pm \sqrt{-}$
25	-12	$\frac{1}{2}r\sqrt{-12}$	$x^3 + \frac{9}{4}r^2x = +\frac{13}{4}r^3$	$\frac{9q}{13p} = \sqrt{\frac{4p}{9}} = \sqrt[3]{\frac{4q}{13}}$	$1 \pm \sqrt{-}$
	$-n$	$\frac{1}{2}r\sqrt{-n}$	$x^3 + \frac{n-3}{4}r^2x = +\frac{n+1}{4}r^3$	$\frac{n-3 \cdot q}{n+1\,p} = \sqrt{\frac{4p}{n-3}} = \sqrt[3]{\frac{4q}{n+1}}$	$1 \pm \sqrt{-n}$

31. Item

31. From the bare inſpection of this table ſeveral uſeful and curious obſervations may be made. And firſt it appears, that when q is poſitive, as in all the forms after the 12th, r is the greateſt root; but when q is negative, or in all the caſes to the 12th, r is one of the leſs roots.

32. In all caſes before the 4th form r is the leaſt root, becauſe $\frac{\sqrt{10}-1}{2}$, or $\frac{\sqrt{11}-1}{2}$, &c. is always greater than 1; and in all ſuch forms $\overline{\frac{1}{2}q}^{2}$ is leſs than $\overline{\frac{1}{3}p}^{3}$; but the former approaches nearer and nearer to an equality with the latter till the 4th form, where $\overline{\frac{1}{2}q}^{2}$ is become $= \overline{\frac{1}{3}p}^{3}$, and r is then equal to one of the other roots, becauſe $\frac{\sqrt{9}-1}{2} = \frac{2}{2} = 1$.

33. From hence r becomes the middle root, and continues ſo to the 12th form, where it becomes equal to what has hitherto been the greateſt root, and the other root becomes at this place $= 0$; and $\overline{\frac{1}{2}q}^{2}$ has decreaſed from the 4th form all the way more and more in reſpect of $\overline{\frac{1}{3}p}^{3}$, till at this 12th form it has become $= 0$, or infinitely leſs than $\overline{\frac{1}{3}p}^{3}$.

34. From this place r becomes the greateſt root, the ſign of q changes to $+$, and $\overline{\frac{1}{2}q}^{3}$ again increaſes in reſpect of $\overline{\frac{1}{3}p}^{3}$, till at the 13th caſe it becomes again equal to it, and the two leſs roots equal to each other, like as at the 4th form.

35. From hence $\overline{\tfrac{1}{2}q}|^2$ becomes greater than $\overline{\tfrac{1}{3}p}|^3$, and increaſes more and more in reſpect of it, till at the 16th ſtep where p is $= 0$, or $\overline{\tfrac{1}{2}q}|^2$ infinitely greater than $\overline{\tfrac{1}{3}p}|^3$.

36. From this place the ſign of p becomes $+$, and $\overline{\tfrac{1}{2}q}|^2$ continually decreaſes in reſpect of $\overline{\tfrac{1}{3}p}|^3$ to infinity.

37. By help of this table we may find the roots of any cubic equation $x^3 \mp px = q$ whenever we can aſſign the relation between $\sqrt{p}$ and $\sqrt[3]{q}$. For ſince one root r is always $= \frac{\overline{n \pm 3} \cdot q}{\overline{n \mp 1} \cdot p} = \sqrt{\frac{4p}{n \pm 3}} = \sqrt[3]{\frac{4q}{n \mp 1}}$, and the other two roots $= -\tfrac{1}{2}r \cdot \overline{1 \pm \sqrt{\pm n}}$, it follows, that if from the equation $\sqrt{\frac{4p}{n \pm 3}} = \sqrt[3]{\frac{4q}{n \mp 1}}$, where the two denominators under the radicals differ by 4, we can aſſign the value of n, the above formula will give us the roots.

38. As if the equation be $x^3 - 18x = -27$. Here $p = 18$, and $q = 27$; then $\sqrt{\frac{4p}{8}} = \sqrt{\frac{p}{2}} = \sqrt{9} = 3$, and $\sqrt[3]{\frac{4q}{4}} = \sqrt[3]{27} = 3$ alſo; therefore $n + 3 = 8$, or $n - 1 = 4$, either of which gives $n = 5$: conſequently, $r = \frac{\overline{n + 3} \cdot q}{\overline{n - 1} \cdot p} = \frac{8q}{4p} = \frac{2q}{p} = \frac{54}{18} = 3$ is the middle root, becauſe $\frac{8q}{4p}$ is found between the 4th and 12th caſes, which are the limits of the middle roots: and $-\tfrac{1}{2}r \cdot \overline{1 \pm \sqrt{n}} = -\tfrac{3}{2} \cdot \overline{1 \pm \sqrt{5}} =$ 4·854102 and 1·854102 are the greateſt and leaſt roots. Or, theſe two roots may be alſo found in the ſame manner

manner from the table of forms, which contains all the roots of every equation, thus: by a few trials I find $\sqrt{\frac{4p}{20 \cdot 95}} = \sqrt[3]{\frac{4q}{16 \cdot 95}}$ nearly, and therefore $\frac{20 \cdot 95q}{16 \cdot 95p} = 1 \cdot 854$ is the leaſt root, becauſe here $n = 17 \cdot 95$ which lies far above the limit for the leaſt roots, which is at the fourth form, where n is $= 9$. And laſtly, $\sqrt{\frac{4p}{3 \cdot 0557}} = \sqrt[3]{\frac{4q}{\cdot 9143}}$ nearly, and therefore, $\frac{3 \cdot 0557q}{\cdot 9143p} = 4 \cdot 854$ is the greateſt root, becauſe $\frac{3 \cdot 0557q}{\cdot 9143p}$ is found between the 12th and 13th forms, which are the limits between which lies the greateſt root of every equation that has all its roots real.

39. Again, let the equation be $x^3 + 2x = 12$. Here $p = 2$, and $q = 12$; hence $\sqrt{\frac{4p}{2}} = \sqrt{2p} = \sqrt{4} = 2$, and $\sqrt[3]{\frac{4q}{6}} = \sqrt[3]{\frac{2}{3}q} = \sqrt[3]{8} = 2$ alſo; therefore $n - 3 = 2$, or $n + 1 = 6$, either of which gives $n = 5$. Conſequently, $r = \frac{\overline{n-3} \cdot q}{\overline{n+1} \cdot p} = \frac{2q}{6p} = \frac{q}{3p} = \frac{12}{6} = 2$, and the other two roots are $-\frac{1}{2}r \cdot \overline{1 \pm \sqrt{-n}} = -1 \cdot \overline{1 \pm \sqrt{-5}} = -1 \mp \sqrt{-5}$.

40. But it is only by trials that we find out a proper value for n in ſuch caſes as theſe; and this is perhaps attended with no leſs trouble than the ſearching out one of the roots by trials from the original cubic equation itſelf. This method of finding the roots would indeed be effectual and ſatisfactory if we had a direct method of

deter-

determining the value of n from the equation $\sqrt{\frac{4p}{n \pm 3}} = \sqrt[3]{\frac{4q}{n \mp 1}}$ by an equation under the 3d degree; but by reducing this equation out of radicals, there refults another cubic equation of no lefs difficulty to refolve than the original one. We muft therefore fearch for other methods of determining the roots; and firft it will be proper to treat of the rule which is called CARDAN'S.

41. Let $x^3 + px = q$ be the general equation where p and q denote any given numbers with their figns, pofitive or negative. And let $z + y$ denote one of the roots of this equation, that is, let the root be divided into any two parts z and y. Hence then $x = z + y$; which value of x being fubftituted for it in the original equation $x^3 + px = q$, that equation will become $z^3 + 3z^2y + 3zy^2 + y^3 + p . \overline{z+y} = q$, or $z^3 + y^3 + 3zy . \overline{z+y} + p . \overline{z+y} = q$. Now on introducing the two unknown quantities z and y, we fuppofed only one condition or equation, namely, $z + y = x$; we are therefore yet at liberty to affume any other poffible condition we pleafe: but this other condition ought to be fuch as will make the equation reducible to a fimple one, or to a quadratic, in order to obtain from it the value of z or y: and for this purpofe there does not feem to be any other proper condition befide

 that

that which ſuppoſes $3zy$ to be $= -p$; and in conſequence of this ſuppoſition, the equation becomes barely $z^3 + y^3 = q$. Now from the ſquare of this equation let four times the cube of $zy = -\frac{1}{3}p$ be ſubtracted, and there will remain $z^6 - 2z^3y^3 + y^6 = q^2 + \frac{4}{27}p^3$, the ſquare root of which is $z^3 - y^3 = \sqrt{q^2 + \frac{4}{27}p^3}$; this laſt being added to, and ſubtracted from, the equation $z^3 + y^3 = q$, we have $\begin{cases} 2z^2 = q + \sqrt{q^2 + \frac{4}{27}p^3} = q + 2\sqrt{\overline{\tfrac{1}{2}q}|^2 + \overline{\tfrac{1}{3}p}|^3}, \\ 2y^3 = q - \sqrt{q^2 + \frac{4}{27}p^3} = q - 2\sqrt{\overline{\tfrac{1}{2}q}|^2 + \overline{\tfrac{1}{3}p}|^3}, \end{cases}$ hence dividing by 2, and extracting the cube roots, we have $\left\{\begin{array}{l} z = \sqrt[3]{\tfrac{1}{2}q + \sqrt{\overline{\tfrac{1}{2}q}|^2 + \overline{\tfrac{1}{3}p}|^3}} \times 1 \text{ or } \times -\frac{1 \pm \sqrt{-3}}{2} \\ y = \sqrt[3]{\tfrac{1}{2}q - \sqrt{\overline{\tfrac{1}{2}q}|^2 + \overline{\tfrac{1}{3}p}|^3}} \times 1 \text{ or } \times -\frac{1 \mp \sqrt{-3}}{2} \end{array}\right\}$ the three values of z and y; for every quantity has three different forms of the cube root, and the cube root of 1, is not only 1, but alſo $-\frac{1 + \sqrt{-3}}{2}$ or $-\frac{1 - \sqrt{-3}}{2}$. Hence then the three values of $z + y$ or x, or the three roots of the equation $x^3 + px = q$, are $\sqrt[3]{\tfrac{1}{2}q + \sqrt{\overline{\tfrac{1}{2}q}|^2 + \overline{\tfrac{1}{3}p}|^3}} \times 1$ or $\times -\frac{1 + \sqrt{-3}}{2}$ or $\times -\frac{1 - \sqrt{-3}}{2} + \sqrt[3]{\tfrac{1}{2}q - \sqrt{\overline{\tfrac{1}{2}q}|^2 + \overline{\tfrac{1}{3}p}|^3}} \times 1$ or $\times -\frac{1 - \sqrt{-3}}{2}$ or $\times -\frac{1 + \sqrt{-3}}{2}$, where the ſigns of $\sqrt{-3}$ muſt be oppoſite in the values of z and y, that is, when it is $\frac{1 \pm \sqrt{-3}}{2}$ in the one, it muſt be $\frac{1 \mp \sqrt{-3}}{2}$ in the other,

otherwiſe

otherwiſe their product zy will not be $= -\frac{1}{3}p$, as it ought to be.

42. Or if we put $a = \frac{1}{3}p$, and $b = \frac{1}{2}q$, the ſame three roots will be

$$\sqrt[3]{b + \sqrt{b^2 + a^3}} + \sqrt[3]{b - \sqrt{b^2 + a^3}} = \text{the 1ſt root or } r,$$

$$-\tfrac{1}{2}\sqrt[3]{b + \sqrt{b^2 + a^3}} \cdot \overline{1 - \sqrt{-3}} - \tfrac{1}{2}\sqrt[3]{b - \sqrt{b^2 + a^3}} \cdot \overline{1 + \sqrt{-3}} \text{ the 2d root.}$$

$$-\tfrac{1}{2}\sqrt[3]{b + \sqrt{b^2 + a^3}} \cdot \overline{1 + \sqrt{-3}} - \tfrac{1}{2}\sqrt[3]{b - \sqrt{b^2 + a^3}} \cdot \overline{1 + \sqrt{-3}} \text{ the 3d root.}$$

43. Or again, the 1ſt root r being

$\sqrt[3]{b + \sqrt{b^2 + a^3}} + \sqrt[3]{b - \sqrt{b^2 + a^3}}$, the other two will be

$$-\tfrac{1}{2}r + \frac{\sqrt{-3}}{2}\sqrt[3]{b + \sqrt{b^2 + a^3}} - \frac{\sqrt{-3}}{2}\sqrt[3]{b - \sqrt{b^2 + a^2}} = \text{the 2d root, and}$$

$$-\tfrac{1}{2}r - \frac{\sqrt{-3}}{2}\sqrt[3]{b + \sqrt{b^2 + a^3}} - \frac{\sqrt{-3}}{2}\sqrt[3]{b - \sqrt{b^2 + a^3}} = \text{the 3d root.}$$

44. Or, if we put $s = \sqrt[3]{b + \sqrt{b^2 + a^3}}$, and $d = \sqrt[3]{b - \sqrt{b^2 + a^3}}$, the roots will be

$$s + d = r \text{ the 1ſt root,}$$

$$-\frac{s + d}{2} + \frac{s - d}{2}\sqrt{-3} = \text{the 2d root,}$$

$$-\frac{s + d}{2} - \frac{s - d}{2}\sqrt{-3} = \text{the 3d root.}$$

45. The firſt of theſe roots x or $r = s + d = \sqrt[3]{b + \sqrt{b^2 + a^3}} + \sqrt[3]{b - \sqrt{b^2 + a^3}}$, is that which is called CARDAN's rule, by whom it was firſt publiſhed, but invented by FERREUS. And this is always a real root, though it is not

 always

always the greateſt root as it has been commonly thought to be.

46. The firſt root $r = s + d = \sqrt[3]{b + \sqrt{b^2 + a^3}} + \sqrt[3]{b - \sqrt{b^2 + a^3}}$, although it be always a real quantity, yet often aſſumes an imaginary form when particular numbers are ſubſtituted inſtead of the letters a and b, or p and q. And this it is evident will happen whenever a is negative and a^3 greater than b^2, or $\overline{\tfrac{1}{3}p}|^3$ greater than $\overline{\tfrac{1}{2}q}|^2$; for then $\sqrt{b^2 + a^3}$ becomes $\sqrt{b^2 - a^3} = \sqrt{\overline{\tfrac{1}{2}q}|^2 - \overline{\tfrac{1}{3}p}|^3}$ the ſquare root of a negative quantity, which is imaginary. And this will evidently happen whenever the equation has three real roots, but at no time elſe, that is in all the firſt 13 caſes of the foregoing table, wherein $\overline{\tfrac{1}{3}p}|^3$ is greater than $\overline{\tfrac{1}{2}q}|^2$, and p negative; the 4th and 13th only excepted, when $\overline{\tfrac{1}{3}p}|^3$ is $= \overline{\tfrac{1}{2}q}|^2$, and therefore $\sqrt{b^2 - a^3} = 0$, and two of the roots become equal, but with contrary ſigns. This root can never aſſume an imaginary form when a or p is poſitive, nor yet when p is negative and $\overline{\tfrac{1}{2}q}|^2$ greater than $\overline{\tfrac{1}{3}p}|^3$; for in both theſe caſes the quantity $\sqrt{b^2 \pm a^3}$ is real, or the ſquare root of a poſitive quantity. And theſe take place after the firſt 13 caſes of the table of forms, that is, in all the caſes which have only one real root. So that this rule of CARDAN's always gives the

root

root in an imaginary form when the equation has no imaginary roots, but in the form of a real quantity when it has imaginary roots.

47. It may, perhaps, ſeem wonderful that CARDAN's theorem ſhould thus exhibit the root of an equation under the form of an imaginary or impoſſible quantity always when the equation has no imaginary roots, but at no time elſe; and it may juſtly be demanded what can be the reaſon of ſo curious an accident. But this ſeeming paradox will be cleared up by the following conſideration. It is plain, that this circumſtance muſt have happened either through ſome impropriety in the manner of deducing the values of z and y from the two aſſumed equations $x = z + y$, and $zy = -\frac{1}{3}p$, or elſe by ſome impoſſibility in one of theſe two conditions themſelves; but, on examination, the deductions are found to be all fairly drawn, and the operations rightly performed. The true cauſe muſt therefore lie concealed in one of theſe two conditions $x = z + y$ and $zy = -\frac{1}{3}p$. In the firſt of them it cannot be, becauſe it only ſuppoſes that a quantity x can be divided into two parts z and y, which is evidently a poſſible ſuppoſition: it can therefore no where exiſt but in the latter, namely, $zy = -\frac{1}{3}p$. Now this ſuppoſition is this, that the product of the two parts z and y, into which the conſtant quantity x is divided, is

equal

equal to $\frac{1}{3}p$ with its fign changed. But this may always take place when p is pofitive; for then $-\frac{1}{3}p$ will be negative, and two numbers, the one pofitive and the other negative, may always be taken fuch that their product fhall be equal to any negative number whatever, and yet their fum be equal to a given quantity x; and this is done by taking the pofitive one as much greater than x as the other is negative; for thus it is evident the pofitive and negative numbers may be increafed without end: wherefore there is no impoffibility in the fuppofition when p is pofitive; and therefore then the formula ought to exhibit only real quantities, that is, in all the cafes after the 16th in the table of forms, as we have before found. But the fame thing cannot always happen when p is negative, or $-\frac{1}{3}p = zy$ is pofitive: for that zy may be pofitive, the figns of the two factors z and y muft be alike, either both + or both −, that is, both + when the fign of x is +, or both − when that is −: but it is well known, that the greateft product which can be made of the two parts into which a conftant quantity x may be divided, is when the parts are equal to each other, or each equal $\frac{1}{2}x$, and therefore the greateft product is equal to $\overline{\frac{1}{2}x}|^2$ or $\frac{1}{4}x^2$: wherefore if $\frac{1}{4}x^2$ be equal to or greater than $-\frac{1}{3}p$, the condition which fuppofes that zy is $= -\frac{1}{3}p$, is poffible, and the formula ought to exprefs the root by real quantities

tities only, otherwiſe not; but $\frac{1}{4}x^2$, or $\frac{1}{4}r^2$, which is the ſame thing, is always leſs than $-\frac{1}{3}p$ in the firſt thirteen caſes of the table of forms; and therefore in all theſe caſes, which are thoſe in which $\overline{\frac{1}{3}p}|^3$ is greater than $\overline{\frac{1}{2}q}|^2$, or all thoſe which have three real roots, the formula ought to exhibit the root with imaginary quantities, as we have before found to happen; the 4th and 13th caſes only excepted, in which $\overline{\frac{1}{3}p}|^3$ is $= \overline{\frac{1}{2}q}|^2$, and therefore the quantity $\sqrt{b^2 - a^3}$ vaniſhes, and two of the roots are equal.

48. Thus then the real cauſe of this circumſtance is made manifeſt, and it is found to be the neceſſary conſequence of the arbitrary hypotheſis which was made, which is found to be poſſible only in certain caſes. So that we cannot expect the formula to exhibit a real quantity in the other caſes, ſince an impoſſible hypotheſis muſt needs lead to an abſurd concluſion.

49. The other two roots $-\frac{s+d}{2} \pm \frac{s-d}{2}\sqrt{-3}$ in their general ſtate appear in an imaginary form; but on the ſubſtitution of numbers for the letters in any example, they come out real or imaginary quantities in thoſe caſes in which they ought to be ſuch. For s being $= g + \sqrt{\mp h}$, and $d = g - \sqrt{\mp h}$ according as the roots are all real or only one is ſuch; and $-\frac{s+d}{2} = -g = -\frac{1}{2}r$ always half the one real root, we ſhall have $\frac{s-d}{2} = \sqrt{\mp h}$ according to

 the

the ſaid two caſes; and conſequently $\frac{s-d}{2}\sqrt{-3} = \sqrt{\pm 3b}$ a real or an imaginary quantity according as the roots are to be real or imaginary.

50. The firſt root r being found from the formula $\sqrt[3]{b + \sqrt{b^2 + a^3}} + \sqrt[3]{b - \sqrt{b^2 + a^3}}$, or by any other means, the other two roots may be exhibited in ſeveral other forms beſides the foregoing, as may be ſhewn in the following manner.

51. The equation being $x^3 + px = q$, and one root r, by ſubſtitution we have $r^3 + pr = q$, and, by ſubtracting, it is - - - $x^3 - r^3 + p \cdot \overline{x - r} = 0$, and, dividing by $x - r$, it becomes $x^2 + rx + r^2 + p = 0$.

Or this ſame equation may be found by barely dividing $x^3 + px - q = 0$ by $x - r = 0$, for the quotient is $x^2 + rx + r^2 + p = 0$. And the reſolution of this quadratic equation gives $x = -\frac{1}{2}r \pm \sqrt{-p - \frac{3}{4}r^2} = -\frac{1}{2}r \pm \frac{1}{2}\sqrt{-4p - 3r^2}$ the other two roots. And from hence again it appears, that theſe two roots are always imaginary when p in the given equation is poſitive; as alſo when it is negative and leſs than $\frac{3}{4}r^2$; which again include all the caſes of the table of forms after the 13th.

52. Again, ſince $r^3 + pr = q$, therefore $r^2 + p = \frac{q}{r}$, and $r^2 = -p + \frac{q}{r}$, and $-3r^2 = 3p - \frac{3q}{r}$; which being

 ſubſtituted

fubftituted in the above value of the two roots, they become $-\frac{1}{2}r \pm \frac{1}{2}\sqrt{-p - \frac{3q}{r}}$.

53. And again, if $-p$ be expelled from this laft form by means of its value $r^2 - \frac{q}{r}$, the fame two roots will be expreffed by $-\frac{1}{2}r \pm \frac{1}{2}\sqrt{r^2 - \frac{4q}{r}} = -\frac{1}{2}r \times \overline{1 \pm \sqrt{1 - \frac{4q}{r^3}}}$.

54. And farther, if r^3 be expelled from this laft form by means of its value $q - pr$, the fame two roots will alfo become $-\frac{1}{2}r \times \overline{1 \pm \sqrt{1 - \frac{4q}{q - pr}}} = -\frac{1}{2}r \times \overline{1 \pm \sqrt{\frac{pr + 3q}{pr - q}}}$.

55. We might have derived the above forms in yet another manner thus. The firft root being r, let the other two roots be v and w: then we fhall have thefe two equations, namely, $v + w = -r$, and $vwr = q$, or $vw = \frac{q}{r}$; from the fquare of the firft of thefe fubtract four times the laft, fo fhall $v^2 - 2vw + w^2 = r^2 - \frac{4q}{r}$; the root of this is $v - w = \sqrt{r^2 - \frac{4q}{r}}$, which being added to, and taken from $v + w = -r$, and dividing by 2, we have $\left.\begin{matrix} v \\ w \end{matrix}\right\} = -\frac{1}{2}r \pm \frac{1}{2}\sqrt{r^2 - \frac{4q}{r}} = -\frac{1}{2}r \times \overline{1 \pm \sqrt{1 - \frac{4q}{r^3}}}$, the fame with one of the formulæ above given; and then by fubftitution the others will be deduced.

56. To illuftrate now the rules $x = s + d$, or $-\frac{s+d}{2} \pm \frac{s-d}{2}\sqrt{-3}$, by fome examples; fuppofe the

 given

given equation to be $x^3 - 36x = 91$. Here $p = -36$, $q = 91$, $a = \frac{1}{3}p = -12$, $b = \frac{91}{2}$; then $c = \sqrt{b^2 + a^3} = \sqrt{\frac{8281}{4} - 1728} = \sqrt{\frac{1369}{4}} = \frac{37}{2}$, $s = \sqrt[3]{b + c} = \sqrt[3]{\frac{91}{2} + \frac{37}{2}} = \sqrt[3]{64} = 4$, and $d = \sqrt[3]{b - c} = \sqrt[3]{\frac{91}{2} - \frac{37}{2}} = \sqrt[3]{27} = 3$. Confequently, $r = s + d = 4 + 3 = 7$ the firft root; and $-\frac{s+d}{2} \pm \frac{s-d}{2}\sqrt{-3} = \frac{-7 \pm \sqrt{-3}}{2}$ the other two roots, which are imaginary.

57. Ex. 2. Let the equation be $x^3 + 30x = 117$. Here $a = \frac{1}{3}p = 10$, $b = \frac{1}{2}q = \frac{117}{2}$; then $c = \sqrt{b^2 + a^3} = \sqrt{\frac{13689}{4} + 1000} = \sqrt{\frac{17689}{4}} = \frac{133}{2}$, $s = \sqrt[3]{b + c} = \sqrt[3]{\frac{117}{2} + \frac{133}{2}} = \sqrt[3]{\frac{250}{2}} = \sqrt[3]{125} = 5$, and $d = \sqrt[3]{b - c} = \sqrt[3]{\frac{117}{2} - \frac{133}{2}} = \sqrt[3]{-\frac{16}{2}} = \sqrt[3]{-8} = -2$. Confequently, $r = s + d = 5 - 2 = 3$ the firft root; and $-\frac{s+d}{2} \pm \frac{s-d}{2}\sqrt{-3} = \frac{-3 \pm 7\sqrt{-3}}{2}$ the other two roots, which are imaginary.

58. Ex. 3. If the equation be $x^3 + 18x = 6$, we fhall have $a = 6$, and $b = 3$; then $c = \sqrt{b^2 + a^3} = \sqrt{9 + 216} = \sqrt{225} = 15$, $s = \sqrt[3]{b + c} = \sqrt[3]{3 + 15} = \sqrt[3]{18}$, and $d = \sqrt[3]{b - c} = \sqrt[3]{3 - 15} = \sqrt[3]{-12} = -\sqrt[3]{12}$. Therefore $r = s + d = \sqrt[3]{18} - \sqrt[3]{12} = \cdot 331313$ the firft root; and $-\frac{s+d}{2} \pm \frac{s-d}{2}\sqrt{-3} = -\frac{\sqrt[3]{18} - \sqrt[3]{12}}{2} \pm \frac{\sqrt[3]{18} + \sqrt[3]{12}}{2}\sqrt{-3}$ the other two roots.

59. Ex. 4. In the equation $x^3 - 15x = 4$, we have $a = -5$, $b = 2$; hence $c = \sqrt{b^2 + a^3} = \sqrt{4 - 125} =$

$\sqrt{-121}$

$\sqrt{-121} = 11\sqrt{-1}$, $s = \sqrt[3]{b + c} = \sqrt[3]{2 + 11\sqrt{-1}} = 2 + \sqrt{-1}$, and $d = \sqrt[3]{b - c} = \sqrt[3]{2 - 11\sqrt{-1}} = 2 - \sqrt{-1}$. Wherefore $r = s + d = 4$ the firſt root; and $-\frac{s+d}{2} \mp \frac{s-d}{2}\sqrt{-3} = -2 \pm \sqrt{-1} \,.\, \sqrt{-3} = -2 \pm \sqrt{3}$ the other two roots, which are alſo real.

60. Ex. 5. The equation $x^3 - 6x = 4$ gives $a = -2$, and $b = 2$; therefore $c = \sqrt{b^2 + a^3} = \sqrt{4 - 8} = \sqrt{-4} = 2\sqrt{-1}$, $s = \sqrt[3]{b + c} = \sqrt[3]{2 + 2\sqrt{-1}} = -1 + \sqrt{-1}$, and $d = \sqrt[3]{b - c} = \sqrt[3]{2 - 2\sqrt{-1}} = -1 - \sqrt{-1}$. And hence $r = s + d = -2$ the firſt root; and $-\frac{s+d}{2} \pm \frac{s-d}{2}\sqrt{-3} = 1 \pm \sqrt{-1} \,.\, \sqrt{-3} = 1 \pm \sqrt{3}$ which are the two extremes, or the greateſt and leaſt roots. So that in this example, CARDAN's rule gives the middle root.

[redacted]

$= \sqrt{-\frac{175}{4}} = \frac{5}{2}\sqrt{-7}$, $s = \sqrt[3]{b+c} = \sqrt[3]{\frac{9}{2} + \frac{5}{2}\sqrt{-7}} =$ $-\frac{3}{2} + \frac{1}{2}\sqrt{-7}$, and $d = \sqrt[3]{b-c} = \sqrt[3]{\frac{9}{2} - \frac{5}{2}\sqrt{-7}} =$ $-\frac{3}{2} - \frac{1}{2}\sqrt{-7}$. Hence $r = s + d = -3$ the middle root; and $-\frac{s+d}{2} \pm \frac{s-d}{2}\sqrt{-3} = \frac{3}{2} \pm \frac{1}{2}\sqrt{-7} \cdot \sqrt{-3} = \frac{3 \pm \sqrt{21}}{2}$ the greateſt and leaſt roots.

63. Ex. 8. Again, from the equation $x^3 - 12x = -8\sqrt{2}$, we have $a = -4$, and $b = -4\sqrt{2}$; hence $c = \sqrt{b^2 + a^3} = \sqrt{32 - 64} = \sqrt{-32} = 4\sqrt{-2}$, $s = \sqrt[3]{b+c}$ $= \sqrt[3]{-4\sqrt{2} + 4\sqrt{-2}} = \sqrt{2} + \sqrt{-2}$, and $d = \sqrt{2} - \sqrt{-2}$. So that $r = s + d = 2\sqrt{2}$ the middle root; and $-\frac{s+d}{2} \pm \frac{s-d}{2}\sqrt{-3} = -\sqrt{2} \pm \sqrt{-2} \cdot \sqrt{-3} =$ $-\sqrt{2} \pm \sqrt{6} = -\sqrt{2} \cdot \overline{1 \mp \sqrt{3}}$ the greateſt and leaſt roots.

64. Ex. 9. But the equation $x^3 - 15x = 22$ gives $a = -5$, and $b = 11$; and therefore $c = \sqrt{b^2 + a^3}$ $= \sqrt{121 - 125} = \sqrt{-4}$, $s = \sqrt[3]{b+c} = \sqrt[3]{11 + \sqrt{-4}} =$ $-1 - \sqrt{-4}$, and $d = -1 + \sqrt{-4}$. Conſequently $r = s + d = -2$ the leaſt root; and $-\frac{s+d}{2} \pm \frac{s-d}{2}\sqrt{-3}$ $= 1 \pm \sqrt{-4} \cdot \sqrt{-3} = 1 \pm \sqrt{12}$ the two greater roots.

65. Ex. 10. Laſtly, in the equation $x^3 - 15x = 20$, we have $a = -5$, and $b = 10$; conſequently $c = \sqrt{b^2 + a^3}$ $= \sqrt{100 - 125} = \sqrt{-25} = 5\sqrt{1}$, $s = \sqrt[3]{b+c} =$ $\sqrt[3]{10 + 5\sqrt{-1}}$, and $d = \sqrt[3]{10 - 5\sqrt{-1}}$. Therefore

$r =$

$x = s + d = \sqrt[3]{10 + 5\sqrt{-1}} + \sqrt[3]{10 - 5\sqrt{-1}} =$ the first root; and $-\frac{s+d}{2} \pm \frac{s-d}{2}\sqrt{-3} = -\frac{\sqrt[3]{10+5\sqrt{-1}} + \sqrt[3]{10-5\sqrt{-1}}}{2}$ $\pm \frac{\sqrt[3]{10+5\sqrt{-1}} - \sqrt[3]{10-5\sqrt{-1}}}{2}\sqrt{-3} =$ the other two roots.

66. Hence it appears, that CARDAN's rule $s + d$ brings out sometimes the greatest root, sometimes the middle root, and sometimes the least root.

Of the Roots by Infinite Series.

67. Another way of assigning the roots of a cubic equation, may be by infinite series, derived from the foregoing formulæ, namely, $s + d$ and $-\frac{s+d}{2} \pm \frac{s-d}{2}\sqrt{-3}$, or $\sqrt[3]{b+c} + \sqrt[3]{b-c}$ and

$-\frac{1}{2} \times \overline{\sqrt[3]{b+c} + \sqrt[3]{b-c}} \pm \frac{1}{2}\sqrt{-3} \times \overline{\sqrt[3]{b+c} - \sqrt[3]{b-c}}.$

For by expanding $\sqrt[3]{b \pm c}$ in an infinite series, we shall evidently have all the roots expressed in such series.

68. Now $s = \sqrt[3]{b+c} = \sqrt[3]{b} \times : 1 + \frac{c}{3b} - \frac{2c^2}{3.6b^2} + \frac{2.5c^3}{3.6.9b^3}$ &c. and $d = \sqrt[3]{b-c} = \sqrt[3]{b} \times : 1 - \frac{c}{3b} - \frac{2c^2}{3.6b^2} - \frac{2.5c^3}{3.6.9b^3}$ &c. Hence $s + d = 2\sqrt[3]{b} \times : 1 - \frac{2c^2}{3.6b^2} - \frac{2.5.8c^4}{3.6.9.12b^4}$ &c. for the first root, as it was found by Mr. NICOLE, in the *Memoires*

Memoires de l'Acad. 1738. Alſo

$s-d=\frac{2c}{\sqrt[3]{b^2}}\times:\frac{1}{3}+\frac{2\cdot 5c^2}{3\cdot 6\cdot 9b^2}+\frac{2\cdot 5\cdot 8\cdot 11c^4}{3\cdot 6\cdot 9\cdot 12\cdot 15b^4}$ &c. Therefore,

$$\left.\begin{array}{l}-\frac{s+d}{2}\\ \pm\frac{s-d}{2}\sqrt{-3}\end{array}\right\}=\left\{\begin{array}{l}-\sqrt[3]{b}\times:1-\frac{2c^2}{3\cdot 6b^2}-\frac{2\cdot 5\cdot 8c^4}{3\cdot 6\cdot 9\cdot 12b^4}\ \&c.\\ \pm\frac{c\sqrt{-3}}{\sqrt[3]{b^2}}\times:\frac{1}{3}+\frac{2\cdot 5c^2}{3\cdot 6\cdot 9b}+\frac{2\cdot 5\cdot 8\cdot 11c^4}{3\cdot 6\cdot 9\cdot 12\cdot 15b^4}\ \&c.\end{array}\right.$$

for the other two roots, which were given by CLAIRAUT, in his *Elemens d'Algebre.*

69. Hence again it appears, that when c^2 is poſitive, theſe two latter roots are imaginary; for then the factor $\frac{c\sqrt{-3}}{\sqrt[3]{b^2}}$ is imaginary. And that thoſe roots are real when this c^2 is negative; for then this factor becomes $\frac{c\sqrt{-1}\times\sqrt{-3}}{\sqrt[3]{b^2}}=\frac{c\sqrt{3}}{\sqrt[3]{b^2}}$, a real quantity. But in this laſt caſe, the ſign of every ſecond term in the two ſeries muſt be changed, namely, the ſigns of the terms containing the odd powers of the negative quantity c^2; for the ſeries contain the letters as adapted to the poſitive ſign only.

70. Theſe ſeries are proper for thoſe caſes only in which c^2 is not greater than b^2; for if c^2 were greater than b^2, they would all diverge, and be of no uſe: and the ſeries proper for the other caſes, namely, in which c^2 is greater than b^2, we ſhall give below.

71. That c^2 be leſs than b^2, or the foregoing ſeries be proper to be uſed, a or $\frac{1}{3}p$ muſt be a negative quantity; for if it be poſitive, then $c^2 = b^2 + a^3$ will be greater than b^2. But for this purpoſe a cannot be *any* negative quantity taken at pleaſure; for if it be ſo taken as that a^3 be greater than $2.b^2$, then ſhall $-c^2 = a^3 - b^2$ be greater than b^2. And hence theſe ſeries converge only in ſome of the caſes of three real roots, and in ſome of thoſe that have only one real root, namely, from the 16th form to ſomewhere between the 12th and 13th forms in the general table Art. 30. when b is poſitive, and conſequently it includes ſome caſes both with and without imaginary roots. But that in all the caſes, the firſt ſeries $s + d = 2\sqrt[3]{b} \times : 1 - \frac{2c^2}{3.6b^2}$ &c. is the greateſt root, as will ſtill more fully appear by conſulting Art. 83.

72. Now, in the firſt place, when $a = 0$, or $c = b$, which is the limit, or 16th caſe in the table Art. 30, the equation being $x^3 = q = 2b$, then the only real root is $s = \sqrt[3]{\overline{b + o}} = \sqrt[3]{2b} = \sqrt[3]{q} = \sqrt[3]{b} \times : 1 + \frac{1}{3} - \frac{2}{3.6} +$ &c. Hence alſo, dividing by $\sqrt[3]{b}$, we have

$$\sqrt[3]{2} = 1 + \frac{1}{3} - \frac{2}{3.6} + \frac{2.5}{3.6.9} - \frac{2.5.8}{3.6.9.12} \text{ \&c.}$$

73. But in this caſe alſo the root is $s + d = 2\sqrt[3]{b} \times : 1 - \frac{2}{3.6} - \frac{2.5.8}{3.6.9.12}$ &c. And

conſequently

consequently this is equal to the former series, or

$$2 \times : 1 - \frac{2}{3.6} - \frac{2.5.8}{3.6.9.12} \text{ \&c.} = 1 + \frac{1}{3} - \frac{2}{3.6} + \frac{2.5}{3.6.9} \text{ \&c.}$$

$= \sqrt[3]{2}$. Hence, by subtracting $1 - \frac{2}{3.6} - \frac{2.5.8}{3.6.9.12}$ &c. from both sides, we have

$$1 - \frac{2}{3.6} - \frac{2.5.8}{3.6.9.12} \text{ \&c.} = \frac{1}{3} + \frac{2.5}{3.6.9} + \frac{2.5.8.11}{3.6.9.12.15} \text{ \&c.}$$

which multiplied by $2\sqrt[3]{b}$ will also give the root of the same equation. And hence, adding $\frac{2}{3.6} + \frac{2.5.8}{3.6.9.12}$ &c. to both sides of the last equation, we find that

$$1 \text{ is } = \frac{1}{3} + \frac{2}{3.6} + \frac{2.5}{3.6.9} + \frac{2.5.8}{3.6.9.12} + \frac{2.5.8.11}{3.6.9.12.15} \text{ \&c.}$$

Or, farther, multiplying by 3, and subtracting 1, we have

$$2 = \frac{2}{6} + \frac{2.5}{6.9} + \frac{2.5.8}{6.9.12} + \frac{2.5.8.11}{6.9.12.15} \text{ \&c.}$$

74. Also from $2 \times : 1 - \frac{2}{3.6} - \frac{2.5.8}{3.6.9.12}$ &c. $= \sqrt[3]{2}$ in the last article, we find $\frac{1}{2}\sqrt[3]{2} =$

$$1 - \frac{2}{3.6} - \frac{2.5.8}{3.6.9.12} \text{ \&c.} = \frac{1}{3} + \frac{2.5}{3.6.9} + \frac{2.5.8.11}{3.6.9.12.15} \text{ \&c.}$$

75. In this case also, namely, $c = b$, the equation $d = \sqrt[3]{b - c} = \sqrt[3]{b} \times : 1 - \frac{c}{3b} - \frac{2c^2}{3.6b^2}$ &c. becomes

$$0 = \sqrt[3]{b} \times : 1 - \frac{1}{3} - \frac{2}{3.6} - \frac{2.5}{3.6.9} \text{ \&c.}$$

And hence, dividing by $\sqrt[3]{b}$, and adding, we have

$$1 = \frac{1}{3} + \frac{2}{3.6} + \frac{2.5}{3.6.9} + \frac{2.5.8}{3.6.9.12} \text{ \&c.}$$

the same as in the last article but one.

76. And

76. And by taking other values of b and c, or other relations between them, any number of infinite ſeries may be aſſigned, whoſe ſums will be given by the two equations $\sqrt[3]{b \pm c} = \sqrt[3]{b} \times : 1 \pm \frac{c}{3b} - \frac{2c^2}{3 \cdot 6b^2} \pm \frac{2 \cdot 5c^3}{3 \cdot 6 \cdot 9b^3}$ &c. And if b be very great in reſpect of c, the two firſt terms of the ſeries will give the cube root true to many places of figures.

77. Hitherto is concerning one of the limits or extreme caſes only, namely, when $c^2 = b^2$, or when the equation is $x^3 = q = 2b$. And it has been obſerved, that the firſt general ſeries for the three roots converges in all the caſes of the equation $x^3 - px = q$, or $x^3 - 3ax = 2b$, in which a^3 is not greater than $2b^2$. But a^3 may be any real quantity not greater than $2b^2$, and ſo it may be either leſs than, equal to, or greater than b^2.

78. When, in this equation, a^3 is leſs than b^2, then c^2 is poſitive, and leſs than b^2, and the firſt ſeries gives the only real root without any change in the ſigns of the terms. And to this belongs all caſes of the equation that can fall in between the 13th and 16th formulæ in the general table in Art. 30.

79. If a^3 be $= b^2$, then $c = 0$, and the three firſt ſeries give $2\sqrt[3]{b} = \sqrt[3]{4q}$ for the greateſt root, and $-\sqrt[3]{b}$ for each of the leſs roots. The ſame as at the 13th form in the general table Art. 30.

80. When a^3 is greater than b^2, c^2 will be negative, and then, changing the ſigns of the odd powers of c^2, the three general ſeries will give the three roots of the equation, which will always be all real. In this claſs are two caſes, namely, when c^2 is leſs than b^2, and when they are equal, which is the limit; for when c^2 becomes greater than b^2, the ſeries diverge.

81. Now when a^3 is between b^2 and $2b^2$, then c^2 is negative and leſs than b^2, and the general ſeries give all the three real roots by changing the ſign of every other term.

82. And when $a^3 = 2b^2$, then $-c^2 = b^2$, and the three roots become thus:

$2\sqrt[3]{b} \times : 1 + \frac{2}{3.6} - \frac{2.5.8}{3.6.9.12} + \frac{2.5.8.11.14}{3.6.9.12.15.18}$ &c. the firſt or greateſt root,

and $\begin{cases} -\sqrt[3]{b} \times : 1 + \frac{2}{3.6} - \frac{2.5.8}{3.6.9.12} + \frac{2.5.8.11.14}{3.6.9.12.15.18} \text{ \&c.} \\ \pm \sqrt[3]{b} \times \sqrt{3} \times : \frac{1}{3} - \frac{2.5}{3.6.9} + \frac{2.5.8.11}{3.6.9.12.15} \text{ \&c.} \end{cases}$

the two leſs roots.

83. The firſt of theſe 3 is the greateſt root, becauſe $\sqrt[3]{b} \times : 1 + \frac{2}{3.6} - \frac{2.5.8}{3.6.9.12}$ &c. is greater than $\sqrt[3]{b} \times \sqrt{3} \times : \frac{1}{3} - \frac{2.5}{3.6.9}$ &c. for $1 + \frac{2}{3.6}$ &c. is greater than 1, and $\sqrt{3} \times : \frac{1}{3} - \frac{2.5}{3.6.9}$ &c. $= \sqrt{\frac{1}{3}} \times : 1 - \frac{2.5}{6.9}$ &c.

is

is less than 1. So that in general the first series gives the greatest of the three roots.

84. But it is evident, that this case agrees with the 10th form in the table Art. 30; in which the middle root r is found to be $\sqrt[3]{\frac{4q}{2}} = \sqrt[3]{2q} = -\sqrt[3]{4b} = -2\sqrt[3]{\frac{1}{2}b}$, and the two other, or greatest and least roots, are $-\frac{1}{2}r \times \overline{1 \pm \sqrt{3}} = \sqrt[3]{\frac{1}{2}b} \times \overline{1 \pm \sqrt{3}}$.

85. Hence by a comparison of these two different forms of the same roots we find

$$\frac{\sqrt{3}+1}{2\sqrt[3]{2}} = 1 + \frac{2}{3.6} - \frac{2.5.8}{3.6.9.12} + \frac{2.5.8.11.14}{3.6.9.12.15.18} \text{ \&c.} = \text{A},$$

and $\frac{\sqrt{3}-1}{2\sqrt[3]{2}} = \frac{1}{3} - \frac{2.5}{3.6.9} + \frac{2.5.8.11}{3.6.9.12.15} - \text{\&c.} = \text{B}$.

86. And by adding and subtracting these two, we find

$$\frac{\sqrt{3}}{\sqrt[3]{2}} = 1 + \frac{1}{3} + \frac{2}{3.6} - \frac{2.5}{3.6.9} - \frac{2.5.8}{3.6.9.12} + + - - \text{\&c. and}$$

$$\frac{1}{\sqrt[3]{2}} = 1 - \frac{1}{3} + \frac{2}{3.6} + \frac{2.5}{3.6.9} - \frac{2.5.8}{3.6.9.12} - + + - - \text{\&c.} = \text{C}.$$

87. Also, because $\frac{\sqrt{3}+1}{2\sqrt[3]{2}} \times \frac{\sqrt{3}-1}{2\sqrt[3]{2}}$ is $= \frac{1}{2\sqrt[3]{4}}$, which is $= \frac{1}{2} \times \overline{\frac{1}{\sqrt[3]{2}}}\Big|^2$; therefore the mean proportional between the two series A and B, is to the series C, as the side of a square is to its diagonal.

88. Moreover, to and from the two series A and B, adding and subtracting the two series in Art. 74. namely,

namely, $\frac{1}{2}\sqrt[3]{2}$ or $\frac{\sqrt[3]{2}}{2} = 1 - \frac{2}{3.6} - \frac{2.5.8}{3.6.9.12}$ &c. $= \frac{1}{3} + \frac{2.5}{3.6.9} + \frac{2.5.8.11}{3.6.9.12.15}$ &c. we obtain the 4 following series:

$$\frac{\sqrt{3}+\sqrt[3]{4}+1}{4\sqrt[3]{2}} = 1 - \frac{2.5.8}{3.6.9.12} - \frac{2.5.8.11.14.17.20}{3.6.9.12.15.18.21.24} \text{ \&c.}$$

$$\frac{\sqrt{3}+\sqrt[3]{4}-1}{4\sqrt[3]{2}} = \frac{1}{3} + \frac{2.5.8.11}{3.6.9.12.15} + \frac{2.5.8.11.14.17.20.23}{3.6.9.12.15.18.21.24.27} \text{ \&c.}$$

$$\frac{\sqrt{3}-\sqrt[3]{4}+1}{4\sqrt[3]{2}} = \frac{2}{3.6} + \frac{2.5.8.11.14}{3.6.9.12.15.18} \text{ \&c.}$$

$$\frac{\sqrt{3}-\sqrt[3]{4}-1}{4\sqrt[3]{2}} = -\frac{2.5}{3.6.9} - \frac{2.5.8.11.14.17}{3.6.9.12.15.18.21} \text{ \&c.}$$

89. It also appears, that the series

$$1 - \frac{1}{3} + \frac{2}{3.6} + \frac{2.5}{3.6.9} - \frac{2.5.8}{3.6.9.12} - \frac{2.5.8.11}{3.6.9.12.15} \text{ \&c.}$$

is the reciprocal of the series

$$1 + \frac{1}{3} - \frac{2}{3.6} + \frac{2.5}{3.6.9} - \frac{2.5.8}{3.6.9.12} + \frac{2.5.8.11}{3.6.9.12.15} \text{ \&c.}$$

where the signs of the former series are found by changing the signs of every other pair of terms in the latter; namely, omitting the first term, change the signs of the 2d and 3d terms, then passing over the 4th and 5th terms, change the signs of the 6th and 7th; and so on. For, by Art 86. the former of these series is equal to $\frac{1}{\sqrt[3]{2}}$; and, by Art. 72. the latter is equal to $\sqrt[3]{2}$.

90. Let

90. Let us now consider the cases in which c^2 is greater than b^2, which include all the cases not comprehended by the former, or in which c^2 is not greater than b^2. And this, it is evident, will happen both when a is positive and when negative; namely when a is any positive quantity whatever, or when it is any negative quantity, and a^3 greater than $2b^2$. And in these two classes, c^2 will be positive or negative, according as a is positive or negative.

91. Now the series in this class will be found the same way as in the last, by only writing here the letter c before the letter b; for then we shall have $s = \sqrt[3]{c+b}$, and $d = \sqrt[3]{-c+b} = -\sqrt[3]{c-b}$.

Then $s = \sqrt[3]{c+b} = \sqrt[3]{c} \times : 1 + \frac{b}{3c} - \frac{2b^2}{3.6c^2} + \frac{2.5b^3}{3.6.9c^3}$ &c.

and $d = -\sqrt[3]{c-b} = \sqrt[3]{c} \times : -1 + \frac{b}{3c} + \frac{2b^2}{3.6c^2} + \frac{2.5b^3}{3.6.9c^3}$ &c.

Hence $s + d = \frac{2b}{\sqrt[3]{c^2}} \times : \frac{1}{3} + \frac{2.5b^2}{3.6.9c^2} + \frac{2.5.8.11b^4}{3.6.9.12.15c^4}$ &c. = the 1st root, and was given by CLAIRAUT. And

$$\left.\begin{array}{l} -\frac{s+d}{2} \\ \pm\frac{s-d}{2}\sqrt{-3} \end{array}\right\} = \left\{\begin{array}{l} \frac{-b}{\sqrt[3]{c^2}} \times : \frac{1}{3} + \frac{2.5b^2}{3.6.9c^2} + \frac{2.5.8.11b^4}{3.6.9.12.15c} \text{ \&c.} \\ \pm\sqrt[3]{c}.\sqrt{-3} \times : 1 - \frac{2b^2}{3.6c^2} - \frac{2.5.8b^4}{3.6.9.12c^4} \text{ \&c.} \end{array}\right.$$

for the other two roots, which, I believe, are new.

92. Here it again appears, that when c^2 is positive, the two latter roots are imaginary; because then

$\sqrt[3]{c} \times \sqrt{-3}$ will be imaginary. But if c^2 be negative, those roots will be both real; since $\sqrt[3]{c} \times \sqrt{-3}$ then becomes $\sqrt[3]{\overline{c \,.\, \sqrt{-1}}} \times \sqrt{-3} = \sqrt[3]{c} \times - \sqrt{-1} \times \sqrt{-3} = - \sqrt[3]{c} \times \sqrt{3}$. The signs prefixed to the terms as above, take place when c^2 is positive; but when c^2 shall be negative, the signs of the terms containing the odd powers of it must be changed. And these series include all the cases in which the former ones failed by not converging. So that between them they comprehend all the cases of the general cubic equation $x^3 \pm px = q$, as they each reciprocally converge when the other diverges, but in no other case, except in the common class, in which c is $= b$, which happens at the two limits, namely, either when a is $= 0$, or when $- a^3 = 2b^2$: and then they both give the same roots. But in the other cases they give the contrary roots; namely, when c is less than b, the first series gives the greatest root; and when c is greater than b, the latter series gives the least root.

93. Now when a is any positive quantity, the first of these series gives the only real root, without any change in the signs of the terms; the other two being imaginary. And this includes all the cases after the 16th in the table in Art. 30.

94. When a is $= 0$, or the limit between positive and negative, as in the 16th form in Art. 30. then is $c = b$,

4 and

and the only real root, or the firft feries, becomes $2\sqrt[3]{b} \times : \frac{1}{3} + \frac{2 . 5}{3 . 6 . 9} +$ &c. which is the fame root as was before found in Art. 73. So that in this 16th cafe, both this feries and the feries in Art. 67. converge, and give the fame and only real root.

95. When a becomes negative, then c^2 becomes negative, and the roots all real. But in this cafe the feries only begins to converge when $-a^3 = 2b^2$, for then $-c^2$ becomes $= b^2$, and then, making the proper change in the figns of the terms, the three roots become

1ft. $-2\sqrt[3]{b} \times : \frac{1}{3} - \frac{2 . 5}{3 . 6 . 9} + \frac{2 . 5 . 8 . 11}{3 . 6 . 9 . 12 . 15}$ &c. the leaft root, and

$$\begin{cases} + \sqrt[3]{b} \times : \frac{1}{3} - \frac{2 . 5}{3 . 6 . 9} + \frac{2 . 5 . 8 . 11}{3 . 6 . 9 . 12 . 15} \text{ \&c.} \\ \pm \sqrt[3]{b} . \sqrt{3} \times : 1 + \frac{2}{3 . 6} - \frac{2 . 5 . 8}{3 . 6 . 9 . 12} + \frac{2 . 5 . 8 . 11 . 14}{3 . 6 . 9 . 12 . 15 . 18} \text{ \&c.} \end{cases}$$

the two greater roots.

96. I have here faid, that the firft of thefe three roots is the leaft of them. To prove which, I affert, that $\sqrt{3} \times : 1 + \frac{2}{3 . 6} - \frac{2 . 5 . 8}{3 . 6 . 9 . 12}$ &c. is greater than 3 times $\frac{1}{3} - \frac{2 . 5}{3 . 6 . 9}$ &c. for $3 \times : \frac{1}{3} - \frac{2 . 5}{3 . 6 . 9}$ &c. $= 1 - \frac{2 . 5}{6 . 9}$ &c. is lefs than 1, whereas $1 + \frac{2}{3 . 6}$ &c. is greater than 1. Confequently, the lefs of the two latter roots, namely,

$\sqrt[3]{b}$

$\sqrt[3]{b} \cdot \sqrt{3} \times : 1 + \frac{2}{3.6}$ &c. $- \sqrt[3]{b} \times : \frac{1}{3} - \frac{2.5}{3.6.9}$ &c. is greater than the first root $2\sqrt[3]{b} \times : \frac{1}{3} - \frac{2.5}{3.6.9}$ &c. That is to say, here the first is the least of the three roots, while in the other class of series the first is the greatest root.

97. Hence, comparing the value of any one of the roots here found, with the value of the same root as found in Art. 82, we obtain the relation between the two series that are concerned in them, namely, that the series $1 + \frac{2}{3.6} - \frac{2.5.8}{3.6.9.12} + \frac{2.5.8.11.14}{3.6.9.12.15.18}$ &c. is to the series $\frac{1}{3} - \frac{2.5}{3.6.9} + \frac{2.5.8.11}{3.6.9.12.15} - \frac{2.5.8.11.14.17}{3.6.9.12.15.18.21}$ &c. as $\sqrt{3} + 1$ is to $\sqrt{3} - 1$, or as $2 + \sqrt{3}$ to 1, or as 1 to $2 - \sqrt{3}$, which are all equal to the same ratio. And the same thing appears from Art. 85.

98. When $-a^3$ becomes greater than $2b^2$, $-c^2$ is greater than b^2, and, by the proper change in the signs, the series for the roots in all cases of this kind become

1st. $\frac{-2b}{\sqrt[3]{c^2}} \times : \frac{1}{3} - \frac{2.5b^2}{3.6.9c^2} + \frac{2.5.8.11b^4}{3.6.9.12.15c^4}$ &c. the least root.

and $\left\{\begin{array}{l} + \frac{b}{\sqrt[3]{c^2}} \times : \frac{1}{3} - \frac{2.5b^2}{3.6.9c^2} + \frac{2.5.8.11b^4}{3.6.9.12.15c^4} \text{ \&c.} \\ \pm \sqrt[3]{c} \cdot \sqrt{3} \times : 1 + \frac{2b^2}{3.6c^2} - \frac{2.5.8b^4}{3.6.9.12c^4} \text{ \&c.} \end{array}\right\}$ the two greater roots.

99. Let us now illustrate all the foregoing series for the roots of cubic equations, by finding by means of them

them the roots of the equations already treated of in Art. 56, &c.

100. And firſt in the equation $x^3 - 36x = 91$. Here $p = -36$, $q = 91$, $a = -12$, $b = 45\frac{1}{2}$, $c^2 = b^2 + a^3$ $= \overline{45\frac{1}{2}}|^2 - 12^3 = 342\frac{1}{4}$, which being poſitive and leſs than b^2, this caſe belongs to the ſeries

$2\sqrt[3]{b} \times : 1 - \frac{2c^2}{3.6b^2} - \frac{2.5.8c^4}{3.6.9.12b^4} -$ &c. in Art. 68.

Now $\frac{c^2}{b^2} = \frac{1369}{8281} = \overline{\frac{37}{91}}|^2 = \cdot 1653182$. Then

$$\begin{array}{lll}
A = & & +\ 1\cdot 0000000 \\
B = \frac{2c^2}{3.6b^2}A & = - & \cdot 0183687 \\
C = \frac{5.8c^2}{9.12b^2}B & = - & 11247 \\
D = \frac{11.14c^2}{15.18b^2}C & = - & 1061 \\
E = \frac{17.20c^2}{21.24b^2}D & = - & 118 \\
F = \frac{23.26c^2}{27.30b^2}E & = - & 14 \\
G = \frac{29.32c^2}{33.36b^2}F & = - & 2 \\
\hline
\text{ſum of the terms} = & & \cdot 9803871 \\
 & & 2 \\
\hline
 & & 1\cdot 9607742 - \text{log. } 0\cdot 2924275 \\
\sqrt[3]{b} = \sqrt[3]{45\cdot 5} & - - - - - & 0\cdot 5526705 \\
\hline
\text{hence the only real root is } 7 & - - - & 0\cdot 8450980
\end{array}$$

That is, $x = 7$ is $= 2\sqrt[3]{\frac{91}{2}} \times : 1 - \frac{2 \cdot 37^2}{3 \cdot 6 \cdot 91^2} - \frac{2 \cdot 5 \cdot 8 \cdot 37^4}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 91^4} - \&c.$

101. The other two roots are imaginary, and in Art. 56 they were found to be $= \frac{-7 \pm \sqrt{-3}}{2}$; but by means of the feries in Art. 68, they are here found to be $\frac{-7}{2} \pm \frac{c\sqrt{-3}}{\sqrt[3]{b^2}} \times : \frac{1}{3} + \frac{2 \cdot 5c^2}{3 \cdot 6 \cdot 9b^2} + \&c.$

Confequently we obtain thefe following fums:

$$\frac{7}{2}\sqrt[3]{\frac{2}{91}} = 1 - \frac{2 \cdot 37^2}{3 \cdot 6 \cdot 91^2} - \frac{2 \cdot 5 \cdot 8 \cdot 37^4}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 91^4} - \frac{2 \cdot 5 \cdot 8 \cdot 11 \cdot 14 \cdot 37^6}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 15 \cdot 18 \cdot 91^6} \&c.$$

$$\frac{1}{37}\sqrt[3]{\frac{91^2}{2^2}} = \frac{1}{3} + \frac{2 \cdot 5 \cdot 37^2}{3 \cdot 6 \cdot 9 \cdot 91^2} + \frac{2 \cdot 5 \cdot 8 \cdot 11 \cdot 37^4}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 15 \cdot 91^4} \&c.$$

102. Ex. 2. In the equation $x^3 + 30x = 117$, we have $a = \frac{1}{3}p = 10$, $b = \frac{1}{2}q = \frac{117}{2} = 58\frac{1}{2}$, and $c^2 = b^2 + a^3 = \overline{\frac{133}{2}}|^2$, which being pofitive, and greater than b^2, the proper feries for this is that in Art. 91, namely,

$$x = \frac{2b}{\sqrt[3]{c^2}} \times : \frac{1}{3} + \frac{2 \cdot 5b^2}{3 \cdot 6 \cdot 9c^2} + \frac{2 \cdot 5 \cdot 8 \cdot 11b^4}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 15c^4} + \&c.$$

Now $\frac{b^2}{c^2} = \overline{\frac{117}{133}}|^2 = \cdot 7738308.$ Hence

A =

$$A = \frac{1}{3} \qquad = {}^{\cdot}333$$

$$B = \frac{2 \cdot 5b^2}{6 \cdot 9c^2} A = \quad 48 \qquad\qquad \frac{2b}{\sqrt[3]{c^2}} = 7{}^{\cdot}128$$

$$C = \frac{8 \cdot 11b^2}{12 \cdot 15c^2} B = \quad 18 \qquad \text{series inverted} \quad 124$$

$$D = \frac{14 \cdot 17b^2}{18 \cdot 21c^2} C = \quad 9 \qquad\qquad 2851$$

$$E = \frac{20 \cdot 23b^2}{24 \cdot 27c^2} D = \quad 5 \qquad\qquad 142$$

$$\qquad\qquad\qquad\qquad 7$$

$$F = \frac{26 \cdot 29b^2}{30 \cdot 33c^2} E = \quad 3 \qquad \text{the root } x = 3{}^{\cdot}000$$

$$G = \frac{32 \cdot 35b^2}{36 \cdot 39c^2} F = \quad 2$$

$$H = \frac{38 \cdot 41b^2}{42 \cdot 45c^2} G = \quad 1$$

$$I = \frac{44 \cdot 47b^2}{48 \cdot 51c^2} H = \quad 1$$

$$K = \frac{50 \cdot 53b^2}{54 \cdot 57c^2} I = \quad 1$$

sum of the terms = $\cdot 421$

That is, $x = 3 = \frac{2 \cdot 117}{\sqrt[3]{2 \cdot 133^2}} \times : \frac{1}{3} + \frac{2 \cdot 5 \cdot 117^2}{3 \cdot 6 \cdot 9 \cdot 133^2} +$ &c.

103. By the other series in the same article the two imaginary roots come out

$= -\frac{3}{2} \pm \sqrt[3]{c} \cdot \sqrt{-3} \times : 1 - \frac{2b^2}{3 \cdot 6c^2} -$ &c. which were before found in Art. 57 to be $-\frac{3}{2} \pm \frac{7}{2}\sqrt{-3}$. Consequently

$$\frac{7}{2}\sqrt[3]{\frac{2}{133}} = 1 - \frac{2 \cdot 117^2}{3 \cdot 6 \cdot 133^2} - \frac{2 \cdot 5 \cdot 8 \cdot 117^4}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 133^4} - \frac{2 \cdot 5 \cdot 8 \cdot 11 \cdot 14 \cdot 117^6}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 15 \cdot 18 \cdot 133^6} \text{ \&c.}$$

$$\frac{1}{39}\sqrt[3]{\frac{133^2}{2^2}} = \frac{1}{3} + \frac{2 \cdot 5 \cdot 117^2}{3 \cdot 6 \cdot 9 \cdot 133^2} + \frac{2 \cdot 5 \cdot 8 \cdot 11 \cdot 117^4}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 15 \cdot 133^4} + \text{ \&c.}$$

104. Ex. 3. In the equation $x^3 + 18x = 6$, we have $a = 6$, $b = 3$, $c = \sqrt{9 + 216} = \sqrt{225} = 15$, real and greater than b, and therefore this caſe belongs to the ſame ſeries as the laſt example. Now $\frac{b^2}{c^2} = \frac{9}{225} = \frac{1}{25} = \cdot 04$, and $\frac{2b}{\sqrt[3]{c^2}} = \frac{6}{\sqrt[3]{225}} = \sqrt[3]{\frac{24}{25}} = \frac{1}{5}\sqrt[3]{120} = \sqrt[3]{\cdot 96}$. Then

$$A = \frac{1}{3} \qquad = \cdot 3333333$$

$$B = \frac{2 \cdot 5b^2}{6 \cdot 9c^2} A \quad = \quad 24692$$

$$C = \frac{8 \cdot 11b^2}{12 \cdot 15c^2} B = \quad 483$$

$$D = \frac{14 \cdot 17b^2}{18 \cdot 21c^2} C = \quad 12$$

$$\cdot 3358520 \quad - \text{ log. } \bar{1}\cdot 5261480$$

$$\sqrt[3]{\cdot 96} \quad - \; - \; - \; - \quad \bar{1}\cdot 9940904$$

$$\text{the root } x = \cdot 3313130 \quad - \; - \quad \bar{1}\cdot 5202384$$

And then the two imaginary roots are

$$-\frac{\cdot 331313}{2} \pm \sqrt[3]{c} \cdot \sqrt{-3} \times : 1 - \frac{2b^2}{3 \cdot 6c^2} \text{ \&c.}$$

105. But, in Art. 58, theſe three roots were found to be $\sqrt[3]{18} - \sqrt[3]{12}$, and $-\frac{\sqrt[3]{18} - \sqrt[3]{12}}{2} \pm \frac{\sqrt[3]{18} + \sqrt[3]{12}}{2}\sqrt{-3}$. Conſequently we have

$$\frac{\sqrt[3]{18} + \sqrt[3]{12}}{2\sqrt[3]{15}} = 1 - \frac{2}{3 \cdot 6 \cdot 25^2} - \frac{2 \cdot 5 \cdot 8}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 25^4} - \text{\&c.}$$

$$\frac{\sqrt[3]{18} - \sqrt[3]{12}}{2}\sqrt[3]{\frac{25}{3}} = \frac{1}{3} + \frac{2 \cdot 5}{3 \cdot 6 \cdot 9 \cdot 25^2} + \frac{2 \cdot 5 \cdot 8 \cdot 11}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 15 \cdot 25^4} + \text{\&c.}$$

106. Ex. 4. In the equation $x^3 - 15x = 4$, we have $a = -5$, $b = 2$, and $c = \sqrt{b^2 + a^3} = \sqrt{-121} = 11\sqrt{-1}$, imaginary and greater than b, which belongs to the ſame ſeries as the laſt 2 examples, but changing the ſign where the odd powers of the negative quantity c^2 is concerned, as in Art. 98.

Now $\frac{b^2}{c^2} = \frac{2^2}{11^2} = \frac{4}{121}$, and $\frac{2b}{\sqrt[3]{c^2}} = \frac{4}{\sqrt[3]{121}} = \sqrt[3]{\frac{64}{121}}$. Then

+	−
$A = \frac{1}{3}$ $= \cdot 3333333$	$B = \frac{2 \cdot 5b^2}{6 \cdot 9c^2} A$ $= \cdot 0020406$
$C = \frac{8 \cdot 11b^2}{12 \cdot 15c^2} B =$ 330	$D = \frac{14 \cdot 17b^2}{18 \cdot 21c^2} C =$ 7
$+ \cdot 3333663$	$- \cdot 0020413$
$- \cdot 0020413$	

the ſeries $= \cdot 3313250$ - log. $\bar{1} \cdot 5202543$

$\sqrt[3]{\frac{64}{121}}$ - - - $\bar{2} \cdot 9077982$

the leaſt root $= - \cdot 2679492$ - - $\bar{1} \cdot 4280525$

107. To find the other roots by this method, we muſt ſum the ſeries $\sqrt[3]{c} \cdot \sqrt{3} \times : 1 + \frac{2b^2}{3 \cdot 6c^2} -$ &c. And as the terms of it are found by multiplying the terms A, B, C, &c. of the former by $\frac{3}{1}$, $\frac{9}{5}$, $\frac{15}{11}$, $\frac{21}{17}$, &c. reſpectively, we ſhall therefore have

$\alpha =$

$\alpha = \frac{3}{1}A \quad = 1$

$\beta = \frac{9}{5}B \quad = 0.0036731 \quad | \quad \gamma = \frac{15}{11}C = -.0000450$

$\delta = \frac{21}{17}D = \quad 8$

$+\ 1.0036739$

$-\ \quad .0000450$

series $= +\ 1.0036289$ - log. 0.0015732

$\sqrt[3]{11}$ - - - - 0.3471309

$\sqrt{3}$ - - - - 0.2385606

$\pm\ 3.8660254$ - 0.5872647

$\frac{1}{2}$ the least root with a contr. sign } $+\ 0.1339746$

sum $+\ 4.0000000$ greatest root

diff. $-\ 3.7320508$ middle root.

108. But the same 3 roots, found in Art. 59, are also 4, and $-2 \pm \sqrt{3}$; which being compared with the series in this example, we find

$$\frac{1+2\sqrt{3}}{2\sqrt[3]{11}} = 1 + \frac{2 \cdot 2^3}{3 \cdot 6 \cdot 11^2} - \frac{2 \cdot 5 \cdot 8 \cdot 2^4}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 11^4} + \frac{2 \cdot 5 \cdot 8 \cdot 11 \cdot 14 \cdot 2^6}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 15 \cdot 18 \cdot 11^6} \&c.$$

$$\frac{2-\sqrt{3}}{4}\sqrt[3]{121} = \frac{1}{3} - \frac{2 \cdot 5 \cdot 2^2}{3 \cdot 6 \cdot 9 \cdot 11^2} + \frac{2 \cdot 5 \cdot 8 \cdot 11 \cdot 2^4}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 15 \cdot 11^4} - \&c.$$

109. Ex. 5. In the equation $x^3 - 6x = 4$, we have $a = -2$, $b = 2$, and $c^2 = b^2 + a^3 = 4 - 8 = -4$, which being negative, and $= b^2$, this case belongs to the series either in Art. 82 or 95. The operation of summing the terms by them is here omitted, because so much room

room would be neceſſary to ſet down ſo great a number of terms, and as the properties ariſing from the ſeries in this caſe have already been noticed above. The 3 roots of this equation have been found in Art. 60 to be -2 and $1 \pm \sqrt{3}$.

110. Ex. 6. In the equation $x^3 - 9x = -10$, we have $a = -3$, $b = -5$, and $c^2 = 25 - 27 = -2$, which being negative and leſs than b^2, the general ſeries in Art. 68, with the neceſſary change of the ſigns, will give the 3 roots. Now $\frac{c^2}{b^2} = \frac{2}{25} = \frac{8}{100} = \cdot 08$, and $\sqrt[3]{b} = -\sqrt[3]{5}$, alſo $\frac{c\sqrt{-3}}{\sqrt[3]{b^2}} = \frac{\sqrt{6}}{\sqrt[3]{25}}$. Hence

$$A = \quad = 1$$
$$B = \frac{2c^2}{3 \cdot 6b^2} A = 0{\cdot}0088889 \qquad C = \frac{5 \cdot 8c^2}{9 \cdot 12b^2} B = {\cdot}0002634$$
$$D = \frac{11 \cdot 14c^2}{15 \cdot 18b^2} C = 120 \qquad E = \frac{17 \cdot 20c^2}{21 \cdot 24b^2} D = 7$$
$$- {\cdot}0002641$$

$$\begin{array}{r} +\ 1{\cdot}0089009 \\ -\ 0{\cdot}0002641 \\ \hline +\ 1{\cdot}0086368 \\ 2 \\ \hline 2{\cdot}0172736 \end{array}$$

$2{\cdot}0172736$ - - log. $0{\cdot}3047649$

$\sqrt[3]{b} = \sqrt[3]{5}$ - - - - $0{\cdot}2329900$

the greateſt root $= -3{\cdot}44948974$ - $0{\cdot}5377549$

111. Then

111. Then for the other roots, by multiplying the terms A, B, C, &c. of the former by $\frac{1}{3}$, $\frac{5}{9}$, $\frac{11}{15}$, &c. we have

$\alpha = \frac{1}{3}A =$	·3333333	$\beta = \frac{5}{9}B =$	·0049383
$\gamma = \frac{11}{15}C =$	1931	$\delta = \frac{17}{21}D =$	97
$\varepsilon = \frac{23}{27}E =$	6		− ·0049480

	+ ·3335270		
	− ·0049480		
	·3285790	log.	$\bar{1}$·5166398
	$\frac{\sqrt{6}}{\sqrt[3]{25}}$		$\bar{1}$·9230956
the second series	± ·27525513		$\bar{1}$·4397354
$\frac{1}{2}$ the greatest root	+ 1·72474487		
middle root	2·00000000		
least root	1·44948974		

112. But, by Art. 61, these 3 roots were found to be 2 and -1 ± 6; which being compared with the series belonging to this case, we find

$$\frac{\sqrt{6}+1}{2\sqrt[3]{5}} = 1 + \frac{2 \cdot 2}{3 \cdot 6 \cdot 25} - \frac{2 \cdot 5 \cdot 8 \cdot 2^2}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 25^2} + \frac{2 \cdot 5 \cdot 8 \cdot 11 \cdot 14 \cdot 2^3}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 15 \cdot 18 \cdot 25^3} \&c.$$

$$\frac{\sqrt{6}-2}{4}\sqrt[3]{25} = \frac{1}{3} - \frac{2 \cdot 5 \cdot 2}{3 \cdot 6 \cdot 9 \cdot 25} + \frac{2 \cdot 5 \cdot 8 \cdot 11 \cdot 2^2}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 15 \cdot 25^2} - \&c.$$

113. Ex. 7. In the equation $x^3 - 12x = 9$, we have $a = -4$, $b = \frac{9}{2}$, and $c^2 = \frac{81}{4} - 64 = -\frac{175}{4}$, which being negative, and greater than b^2, we shall have 3 real roots by the series in Art. 98.

Now $\frac{b^2}{c^2} = \frac{81}{175}$, $\frac{b}{\sqrt[3]{c^2}} = \frac{9}{\sqrt[3]{350}} = \sqrt[3]{\frac{729}{350}}$, and

$\sqrt[3]{c} = \sqrt[6]{\frac{175}{4}} = \sqrt[6]{43{\cdot}75}$. Then

$A = \frac{1}{3}$	$= {\cdot}33333$	$B = \frac{2 \cdot 5 b^2}{6 \cdot 9 c^2} A$	$= {\cdot}02857$
$C = \frac{8 \cdot 11 b^2}{12 \cdot 15 c^2} B =$	647	$D = \frac{14 \cdot 17 b^2}{18 \cdot 21 c^2} C =$	188
$E = \frac{20 \cdot 23 b^2}{24 \cdot 27 c^2} D =$	62	$F = \frac{26 \cdot 29 b^2}{30 \cdot 33 c^2} E =$	22
$G = \frac{32 \cdot 35 b^2}{36 \cdot 39 c^2} F =$	8	$H = \frac{38 \cdot 41 b^2}{42 \cdot 45 c^2} G =$	3
$I = \frac{44 \cdot 47 b^2}{48 \cdot 51 c^2} H =$	1	$K = \frac{50 \cdot 53 b^2}{54 \cdot 57 c^2} I =$	1
	$+ {\cdot}34051$		$- {\cdot}03071$

$$\begin{array}{rlr} + {\cdot}34051 & & \\ - {\cdot}03071 & & \\ \hline {\cdot}30980 & & \\ 2 & & \\ \hline {\cdot}61960 & \text{- - log.} & \bar{1}{\cdot}7921114 \\ \sqrt[3]{\frac{729}{350}} & \text{- - -} & 0{\cdot}1062198 \\ \hline \end{array}$$

the least root $= - {\cdot}79128$ - - $\bar{1}{\cdot}8983312$

114. Then, since the terms of the latter series are found by multiplying the terms of the former by the fractions $\frac{3}{1}$, $\frac{9}{5}$, $\frac{15}{11}$, $\frac{21}{17}$, &c. they will be thus:

$\alpha = \frac{3}{1}A = 1{\cdot}00000$	
$\beta = \frac{9}{5}B = \quad 5143$	$\gamma = \frac{15}{11}C = {\cdot}00882$
$\delta = \frac{21}{17}D = \quad 232$	$\varepsilon = \frac{27}{23}E = \quad 73$
$\zeta = \frac{33}{29}F = \quad 25$	$\eta = \frac{39}{35}G = \quad 9$
$\theta = \frac{45}{41}H = \quad 4$	$\iota = \frac{51}{47}I = \quad 1$
	$- {\cdot}00965$

$+ 1{\cdot}05404$
$- 0{\cdot}00965$

$1{\cdot}04439$	- - - log.	$0{\cdot}0188627$
$\sqrt[6]{43{\cdot}75}$	- - - - - -	$0{\cdot}2734963$
$\sqrt{3}$	- - - - - - -	$0{\cdot}2385606$
last series $\pm 3{\cdot}39564$	- -	$0{\cdot}5309196$
$-\frac{1}{2}$ the first $+ 0{\cdot}39564$		
greatest root $+ 3{\cdot}79128$		
middle root $- 3{\cdot}00000$		

115. But, by Art. 62, these same 3 roots are, -3, and $\frac{3 \pm \sqrt{21}}{2}$; which being compared with the series belonging to this case, we find

$$\frac{\sqrt{21}+9}{12\sqrt[6]{350}}\sqrt{6} = 1 + \frac{2 \cdot 81}{3 \cdot 6 \cdot 175} - \frac{2 \cdot 5 \cdot 8 \cdot 81^2}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 175^2} + \&c.$$

$$\frac{\sqrt{21}-3}{36}\sqrt[3]{350} = \frac{1}{3} - \frac{2 \cdot 5 \cdot 81}{3 \cdot 6 \cdot 9 \cdot 175} + \frac{2 \cdot 5 \cdot 8 \cdot 11 \cdot 81^2}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 15 \cdot 175^2} - \&c.$$

116. Ex. 8. In the equation $x^3 - 12x = -8\sqrt{2}$, we have $a = -4$, $b = -4\sqrt{2}$, and $c^2 = 32 - 64 = -32$, which being negative, and equal to b^2, the 3 roots will

be

be found, by both the forms of series, like as in Ex. 5, Art. 109; but the operation is here omitted for the same reasons as were there given. The 3 roots of this equation were, in Art. 63, found to be $2\sqrt{2}$ and $-\sqrt{2}\pm\sqrt{6}$.

117. Ex. 9. In the equation $x^3 - 15x = 22$, we have $a = -5$, $b = 11$, and $c^2 = 121 - 125 = -4$, which being negative, and less than b^2, the series in Art. 68 give these 3 roots:

Greatest root $= 2\sqrt[3]{11} \times : 1 + \frac{2c^2}{3 \cdot 6b^2} - \frac{2 \cdot 5 \cdot 8c^4}{3 \cdot 6 \cdot 9 \cdot 12b^4}$ &c.

The two less roots $\left\{ \begin{array}{l} -\sqrt[3]{11} \times : 1 + \frac{2c^2}{3 \cdot 6b^2} - \frac{2 \cdot 5 \cdot 8c^4}{3 \cdot 6 \cdot 9 \cdot 12b^4} \text{ \&c.} \\ \pm \frac{2\sqrt{3}}{\sqrt[3]{121}} \times : \frac{1}{3} - \frac{2 \cdot 5c^2}{3 \cdot 6 \cdot 9b^2} + \frac{2 \cdot 5 \cdot 8 \cdot 11c^4}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 15b^4} \text{ \&c.} \end{array} \right\}$ where $\frac{c^2}{b^2} = \frac{4}{121}$

Here

$A = \quad = 1{\cdot}0000000$

$B = \frac{2c^2}{3 \cdot 6b^2} A \quad = \quad 36731 \quad \Big| \quad C = \frac{5 \cdot 8c^2}{9 \cdot 12b^2} B = -{\cdot}0000450$

$D = \frac{11 \cdot 14c^2}{15 \cdot 18b^2} C = \quad 8$

$+\ 1{\cdot}0036739$
$-\ 0{\cdot}0000450$

$1{\cdot}0036289$
2

$2{\cdot}0072578$ - - log. $0{\cdot}3026031$
$\sqrt[3]{11}$ - - - $0{\cdot}3471309$

the greatest root $= 4{\cdot}4641016$ - - $0{\cdot}6497340$

 118. Again,

118. Again,

$\alpha = \frac{1}{3}A =$	$\cdot 3333333$	$\beta = \frac{5}{9}B =$	$\cdot 0020406$
$\gamma = \frac{11}{15}C =$	330	$\delta = \frac{17}{21}D =$	7
	$+ \cdot 3333663$		$- \cdot 0020413$
	$- \cdot 0020413$		
	$\cdot 3313250$		
	2		
	$\cdot 6626500$	log.	$\bar{1}\cdot 8212842$
	$\sqrt{3}$		$0\cdot 2385606$
	$\sqrt[3]{121}$		$- 0\cdot 6942618$
the latter series $\pm$	$\cdot 2320508$		$\bar{1}\cdot 3655830$
$\frac{1}{2}$ the first $-$	$2\cdot 2320508$		
middle root $= -$	$2\cdot 4641016$		
least root $= -$	$2\cdot 0000000$		

119. But, by Art. 64, the 3 roots are -2 and $1 \pm \sqrt{12}$; hence

$$\frac{1+2\sqrt{3}}{2\sqrt[3]{11}} = 1 + \frac{2 \cdot 2^2}{3 \cdot 6 \cdot 11^2} - \frac{2 \cdot 5 \cdot 8 \cdot 2^4}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 11^4} + \frac{2 \cdot 5 \cdot 8 \cdot 11 \cdot 14 \cdot 2^6}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 15 \cdot 18 \cdot 11^6} \&c.$$

$$\frac{2-\sqrt{3}}{4}\sqrt[3]{121} = \frac{1}{3} - \frac{2 \cdot 5 \cdot 2^2}{3 \cdot 6 \cdot 9 \cdot 11^2} + \frac{2 \cdot 5 \cdot 8 \cdot 11 \cdot 2^4}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 15 \cdot 11^4} - \&c.$$

120. And in this manner the roots of cubic equations may always be found by these series; and then by comparing them with the roots of the same equations, as found by other methods, we shall obtain as many series as we please, whose sums will be given.

121. Hence

121. Hence alſo we may find the ſum of any general ſeries of either of theſe forms, namely,

$1 \mp \frac{2g^2}{3.6} - \frac{2.5.8g^4}{3.6.9.12} \mp \frac{2.5.8.11.14g^6}{3.6.9.12.15.18}$ &c. or

$\frac{1}{3} \pm \frac{2.5g^2}{3.6.9} + \frac{2.5.8.11g^4}{3.6.9.12.15} \pm \frac{2.5.8.11.14.17g^6}{3.6.9.12.15.18.21}$ &c. by comparing them with the roots of given cubic equations; whatever be the value of g, not greater than 1.

122. For, by Art. 68, $\sqrt[3]{b+c} + \sqrt[3]{b-c} = 2\sqrt[3]{b} \times :$ $1 - \frac{2c^2}{3.6b^2} - \frac{2.5.8c^4}{3.6.9.12b^4}$ &c. is = the greateſt root of the cubic equation $x^3 - 3\sqrt[3]{b^2-c^2} . x = 2b$. Now make $2\sqrt[3]{b} = 1$, and $\frac{c^2}{b^2} = g^2$; ſo ſhall the above become $\frac{1}{2}\sqrt[3]{1+g} + \frac{1}{2}\sqrt[3]{1-g} = 1 - \frac{2g^2}{3.6} - \frac{2.5.8g^4}{3.6.9.12}$ &c. = the greateſt root of the equation $x^3 - \frac{3}{4}\sqrt[3]{1-g^2} . x = \frac{1}{4}$. And when g^2 or $\frac{c^2}{b^2}$ is negative, theſe become

$\frac{1}{2}\sqrt[3]{1+g\sqrt{-1}} + \frac{1}{2}\sqrt[3]{1-g\sqrt{-1}} = 1 + \frac{2g^2}{3.6} - \frac{2.5.8g^4}{3.6.9.12} +$ &c.

= the greateſt root of the equation $x^3 - \frac{3}{4}\sqrt[3]{1+g^2} . x = \frac{1}{4}$. So that in general the infinite ſeries

$1 \mp \frac{2g^2}{3.6} - \frac{2.5.8g^4}{3.6.9.12} \mp \frac{2.5.8.11.14g^6}{3.6.9.12.15.18}$ &c. is

$= \frac{1}{2}\sqrt[3]{1+g\sqrt{\pm 1}} + \frac{1}{2}\sqrt[3]{1-g\sqrt{\pm 1}}$ = the greateſt root of the equation $x^3 - \frac{3}{4}\sqrt[3]{1 \mp g^2} . x = \frac{1}{4}$. Where the upper and under ſigns reſpectively correſpond to each other.

123. Again,

123. Again,

$$\sqrt[3]{c+b} - \sqrt[3]{c-b} = \frac{2b}{\sqrt[3]{c^2}} \times : \frac{1}{3} + \frac{2 \cdot 5 b^2}{3 \cdot 6 \cdot 9 c^2} + \frac{2 \cdot 5 \cdot 8 \cdot 11 b^4}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 15 c^4} \&c.$$

is the leaſt root of the equation $x^3 + 3\sqrt[3]{c^2-b^2} \cdot x = 2b$. Then, by taking $\frac{2b}{\sqrt[3]{c^2}} = 1$, and $\frac{b^2}{c^2} = g^2$, this becomes

$$\frac{\sqrt[3]{1+g} - \sqrt[3]{1-g}}{2g} = \frac{1}{3} + \frac{2 \cdot 5 g^2}{3 \cdot 6 \cdot 9} \&c.$$ = the leaſt root of the equation $x^3 + \frac{3\sqrt[3]{1-g^2}}{4g^2} x = \frac{1}{4g^2}$. And when g^2 or c^2 is negative, this becomes

$$\frac{\sqrt[3]{1+g\sqrt{-1}} - \sqrt[3]{1-g\sqrt{-1}}}{2g\sqrt{-1}} = \frac{1}{3} - \frac{2 \cdot 5 g^2}{3 \cdot 6 \cdot 9} + \&c.$$ = the leaſt root of the equation $x^3 - \frac{3\sqrt[3]{1+g^2}}{4g^2} x = \frac{-1}{4g^2}$. So that in general the infinite ſeries

$$\frac{1}{3} \pm \frac{2 \cdot 5 g^2}{3 \cdot 6 \cdot 9} + \frac{2 \cdot 5 \cdot 8 \cdot 11 g^4}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 15} \pm \frac{2 \cdot 5 \cdot 8 \cdot 11 \cdot 14 \cdot 17 g^6}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 15 \cdot 18 \cdot 21} \&c. \text{ is}$$

$$= \frac{\sqrt[3]{1+g\sqrt{\pm 1}} - \sqrt[3]{1-g\sqrt{\pm 1}}}{2g\sqrt{\pm 1}}$$ = the leaſt root of the equation $x^3 \pm \frac{3\sqrt[3]{1 \mp g^2}}{4g^2} x = \frac{\pm 1}{4g^2}$.

Of the Roots by another Claſs of Series.

124. But there are yet other ſeries, converging much faſter than thoſe in the foregoing claſs, by the help of which, and CARDAN's rule conjointly, may always be found

found the roots of thofe equations in which that rule fails when it is applied fingly, that is, in what is called the irreducible cafe, or that in which c^2 is negative. And thofe feries are found by introducing another cubic equation having the fame values of b and c^2 as the given equation, except that in the new equation the value of c^2 is pofitive, while in the given one it is negative. For when c^2 is pofitive, the new equation to which it belongs has only one real root, and that root is always found by CARDAN's rule; but the contrary takes place when c^2 is negative, the equation having then three real roots, although they are not always determinable by that rule, becaufe the radical quantities can feldom be extracted, on account of the fquare root of the negative quantity which is contained in them.

125. Now the general expreffion for the root by CARDAN's rule being $s + d = \sqrt[3]{b + \sqrt{\pm c^2}} + \sqrt[3]{b - \sqrt{\pm c^2}}$ or $\sqrt[3]{\sqrt{\pm c^2} + b} - \sqrt[3]{\sqrt{\pm c^2} - b}$, if the cubic roots of each of thefe be extracted by the binomial theorem, as at Art. 68, we fhall obtain thefe 4 forms;

1. $\sqrt[3]{b + \sqrt{+c^2}} + \sqrt[3]{b - \sqrt{+c^2}} = 2\sqrt[3]{b} \times : 1 - \frac{2c^2}{3 \cdot 6b^2} - \&c.$

2. $\sqrt[3]{b + \sqrt{-c^2}} + \sqrt[3]{b - \sqrt{-c^2}} = 2\sqrt[3]{b} \times : 1 + \frac{2c^2}{3 \cdot 6b^2} - \&c.$

3. $\sqrt[3]{\sqrt{+c^2} + b} - \sqrt[3]{\sqrt{+c^2} - b} = \frac{2b}{\sqrt[3]{c^2}} \times : \frac{1}{3} + \frac{2 \cdot 5b^2}{3 \cdot 6 \cdot 9c^2} + \&c.$

4. $\sqrt[3]{\sqrt{-c^2} + b} - \sqrt[3]{\sqrt{-c^2} - b} = \frac{2b}{\sqrt[3]{c^2}} \times : -\frac{1}{3} + \frac{2 \cdot 5b^2}{3 \cdot 6 \cdot 9c^2} - \&c.$

126. Of

126. Of which the series in the first and third denote the only real root of the equation when c^2 is positive, according as c is greater or less than b, which root call X; and the series in the second and fourth forms denote the greatest and least roots of the equation when c^2 is negative, which roots call R and r respectively. Then by adding and subtracting the first and second, as also the third and fourth, there result these four equations;

$$R + X = 4\sqrt[3]{b} \times : 1 - \frac{2.5.8c^4}{3.6.9.12b^4} - \frac{2.5.8.11.14.17.20c^8}{3.6.9.12.15.18.21.24b^8} \&c.$$

$$R - X = 4\sqrt[3]{b} \times : \frac{2c^2}{3.6b^2} + \frac{2.5.8.11.14c^6}{3.6.9.12.15.18b^6} + \&c.$$

$$X - r = \frac{4b}{\sqrt[3]{c^2}} \times : \frac{1}{3} + \frac{2.5.8.11b^4}{3.6.9.12.15c^4} + \&c.$$

$$X + r = \frac{4b}{\sqrt[3]{c^2}} \times : \frac{2.5b^2}{3.6.9c^2} + \frac{2.5.8.11.14.17b^6}{3.6.9.12.15.18.21c^6} + \&c.$$

127. And hence, by equal addition or subtraction, we find these two different expressions both for the greatest and least roots of a cubic equation in which c^2 or $b^2 + a^3$ is negative, namely,

$$R = -X + 4\sqrt[3]{b} \times : 1 - \frac{2.5.8c^4}{3.6.9.12b^4} - \frac{2.5.8.11.14.17.20c^8}{3.6.9.12.15.18.21.24b^8} \&c. \text{ or}$$

$$R = X + 4\sqrt[3]{b} \times : \frac{2c^2}{3.6b^2} + \frac{2.5.8.11.14c^6}{3.6.9.12.15.18b^6} + \&c.$$

$$r = X - \frac{4b}{\sqrt[3]{c^2}} \times : \frac{1}{3} + \frac{2.5.8.11b^4}{3.6.9.12.15c^4} + \frac{2.5.8.11.14.17.20.23b^8}{3.6.9.12.15.18.21.24.27c^8} \&c. \text{ or}$$

$$r = -X + \frac{4b}{\sqrt[3]{c^2}} \times : \frac{2.5b^2}{3.6.9c^2} + \frac{2.5.8.11.14.17b^6}{3.6.9.12.15.18.21c^6} + \&c.$$

where R is the greatest, and r the least root of the equa-

I tion

tion $x^3 - 3ax = 2b$ or $x^3 - 3\sqrt[3]{c^2+b^2} . x = 2b$, and x the only real root of the equation $x^3 + 3\sqrt[3]{c^2-b^2} . x = 2b$; in which, as well as in the above ſeries, c^2 denotes a poſitive quantity.

128. And hence it can no longer be ſaid that CARDAN's rule is of no uſe in the ſolution of cubic equations that have three real roots; ſince they have here been reduced to the other caſe in which the equation has but one real root, which caſe is always reſolvable by that rule. And the firſt hint of ſuch reduction I received from FRANCIS MASERES, Eſq. Curſitor Baron of the Exchequer, he having done me the favour to communicate to me the ſecond of the above four forms for the greateſt root, in a letter of the 17th of July 1779; the inveſtigation of which formula, together with thoſe of the other three, nearly as above, I had the honour of ſending him in a letter of the 26th of the ſame month; and that learned gentleman has ſince communicated to the Royal Society his ſaid formula, together with his own inveſtigation of it, done in his uſual very accurate manner. Since that time I have ſeen, in the *Memoires de l'Acad.* for the year 1743, four expreſſions ſimilar to the above, given by Mr. NICOLE for the purpoſe of ſumming certain terms of a binomial raiſed to

 any

any power, but unaccompanied with any appearance of the idea of thus reducing the one cafe of the cubic equation to the other.

129. It is hardly neceffary to remark, that any general feries of each of the above four forms, is fummed by means of the fum or difference of the roots of thefe two equations $x^3 - 3\sqrt[3]{b^2 \pm c^2} \cdot x = 2b$, and that by fubftituting particular numbers for b and c, we may thus fum as many feries of thofe forms as we pleafe.

130. Ex. 1. We may now illuftrate thefe formulas by fome examples. And firft in the equation $x^3 - 15x = 4$. Here $2b = 4$, and $3\sqrt[3]{b^2 + c^2} = 15$, confequently $b = 2$, and $c^2 = 5^3 - b^2 = 125 - 4 = 121 = 11^2$, and $x = \sqrt[3]{c+b} - \sqrt[3]{c-b} = \sqrt[3]{13} - \sqrt[3]{9} = \cdot 2712508$ the root of the equation $x^3 - 3\sqrt[3]{b^2 - c^2} \cdot x = 2b$ or $x^3 + 3\sqrt[3]{117} \cdot x = 4$. And as b is lefs than c, this equation belongs to the two feries in the latter cafe for finding the leaft root. Hence, the terms of the two feries agreeing with the pofitive and negative terms of the feries in Art. 106, they will ftand thus:

By

By the 1st series			By the 2d series		
$A = \cdot 3333333$			$B = \cdot 0020406$		
$C = \cdot 0000330$			$D = \cdot 0000007$		
$\cdot 3333663$	log.	$\bar{1}\cdot 5229217$	$\cdot 0020413$	log.	$\bar{3}\cdot 3099068$
$\frac{4b}{\sqrt[3]{c^2}} = \sqrt[3]{\frac{512}{121}}$		$0\cdot 2088282$	$\frac{4b}{\sqrt[3]{c^2}} = \sqrt[3]{\frac{512}{121}}$		$0\cdot 2088282$
series $= - \cdot 5392000$		$\bar{1}\cdot 7317499$	series $= + \cdot 0033016$		$\bar{3}\cdot 5187350$
$X = + \cdot 2712508$			$X = - \cdot 2712508$		
$r = - \cdot 2679492$ the least root			$r = - \cdot 2679492$ the same root.		

Agreeing with the same root found in Ex. 4. Art. 106.

131. But the same root has been found to be $-2+\sqrt{3}$ in Art. 59, and hence we obtain the sums of these two particular series, thus,

$$\frac{\sqrt[3]{13}-\sqrt[3]{9}+2-\sqrt{3}}{8}\sqrt[3]{121} = \frac{1}{3} + \frac{2.5.8.11.2^4}{3.6.9.12.15.11^4} + \&c.$$

$$\frac{\sqrt[3]{13}-\sqrt[3]{9}-2+\sqrt{3}}{8}\sqrt[3]{121} = \frac{2.5.2^2}{3.6.9.11^2} + \frac{2.5.8.11.14.17.2^6}{3.6.9.12.15.18.21.11^6}\&c.$$

132. Also by taking the sum and difference of these two, we have

$$\frac{\sqrt[3]{13}-\sqrt[3]{9}}{4}\sqrt[3]{121} = \frac{1}{3} + \frac{2.5.2^2}{3.6.9.11^2} + \frac{2.5.8.11.2^4}{3.6.9.12.15.11^4} + \&c.$$

$$\frac{2-\sqrt{3}}{4}\sqrt[3]{121} = \frac{1}{3} - \frac{2.5.2^2}{3.6.9.11^2} + \frac{2.5.8.11.2^4}{3.6.9.12.15.11^4} - \&c.$$

And this last expression agrees with what was found in Art. 108.

133. Ex. 2. Again in the equation $x^3 - 9x = -10$, we have $2b = -10$, and $3\sqrt[3]{b^2 + c^2} = 9$; conſequently $b = -5$, and $c^2 = 3^3 - b^2 = 27 - 25 = 2$, which being leſs than b^2 or 25, this equation belongs to the firſt claſs of ſeries, or that for the greateſt root. Now

$$x = \sqrt[3]{b + c} + \sqrt[3]{b - c} = \sqrt[3]{-5 + \sqrt{2}} + \sqrt[3]{-5 - \sqrt{2}}$$
$$= -\sqrt[3]{5 - \sqrt{2}} - \sqrt[3]{5 + \sqrt{2}}$$
$$= -\sqrt[3]{3 \cdot 58578864} - \sqrt[3]{6 \cdot 41421356}$$
$$= -1 \cdot 530600 - 1 \cdot 858009 = -3 \cdot 388609$$

= the root of the equation $x^3 - 3\sqrt[3]{b^2 - c^2} . x = 2b$, or $x^3 - 3\sqrt[3]{21} . x = -10$. And the terms of the two ſeries are found as in Art. 110, namely $1 - \frac{2.5.8c^4}{3.6.9.12b^4}$ − &c. $= \text{A} - \text{C} - \text{E} -$ &c. $= \cdot 9997359$, and $\frac{2c^2}{3.6b^2}$ + &c. = B + D + &c. $= \cdot 0089009$. Alſo $4\sqrt[3]{b} = 4\sqrt[3]{-5} = -4\sqrt[3]{5} = -\sqrt[3]{320}$. Then

By the 1ſt ſeries			By the 2d ſeries		
	·9997359	log. $\bar{1}$·9998854		·0089009	log. $\bar{3}$·9494339
	$-\sqrt[3]{320}$	0·8350500		$-\sqrt[3]{320}$	0·8350500
ſeries	= − 6·838098	0·8349354	ſeries	= − ·060881	$\bar{2}$·7844839
X	= + 3·388609		X	= − 3·388609	
	− 3·449489 the greateſt root			− 3·449490 the ſame root.	

And theſe values of the greateſt root are nearly the ſame with that found in Art. 110.

134. But

134. But in Art. 61, the ſame root was found to be $-1-\sqrt{6}$, hence we obtain the ſums of theſe firſt two particular ſeries; and by the addition and ſubtraction of theſe two ariſe the other two following them, namely,

$$\frac{1+\sqrt{6}+\sqrt[3]{5+\sqrt{2}}+\sqrt[3]{5-\sqrt{2}}}{4\sqrt[3]{5}} = 1 - \frac{2.5.8.2^2}{3.6.9.12.5^4} - \&c.$$

$$\frac{1+\sqrt{6}-\sqrt[3]{5+\sqrt{2}}-\sqrt[3]{5-\sqrt{2}}}{4\sqrt[3]{5}} = \frac{2.2}{3.6.5^2} + \frac{2.5.8.11.14.2^3}{3.6.9.12.15.18.5^6} + \&c.$$

$$\frac{1+\sqrt{6}}{2\sqrt[3]{5}} = 1 + \frac{2.2}{3.6.5^2} - \frac{2.5.8.2^2}{3.6.9.12.5^4} + \frac{2.5.8.11.14.2^3}{3.6.9.12.15.18.5^6} - \&c.$$

$$\frac{\sqrt[3]{5+\sqrt{2}}+\sqrt[3]{5-\sqrt{2}}}{2\sqrt[3]{5}} = 1 - \frac{2.2}{3.6.5^2} - \frac{2.5.8.2^2}{3.6.9.12.5^4} - \&c.$$

And the laſt but one of theſe equations agrees with one found in Art. 112.

135. Ex. 3. Alſo in the equation $x^3 - 12x = 9$, we have $2b = 9$, and $\sqrt[3]{b^2+c^2} = 4$; conſequently $b = \frac{9}{2}$, and $c^2 = 4^3 - b^2 = 64 - \frac{81}{4} = \frac{175}{4}$, which being greater than b^2 or $\frac{81}{4}$, this caſe belongs to the ſecond claſs of ſeries, or that of the leaſt roots. Now here $x = \sqrt[3]{c+b} - \sqrt[3]{c-b} = \sqrt[3]{\frac{\sqrt{175}+9}{2}} - \sqrt[3]{\frac{\sqrt{175}-9}{2}} = \sqrt[3]{11{\cdot}114378} - \sqrt[3]{2{\cdot}114378} = 2{\cdot}2316619 - 1{\cdot}2834950 = {\cdot}9481669 =$ the root of the equation $x^3 - 3\sqrt[3]{b^2-c^2}\,.\,x = 2b$ or $x^3 + 3\sqrt[3]{\frac{47}{2}}\,.\,x = 9$. And the terms of the two ſeries being found as in Art. 113, namely, A + C + E + &c. = ·34051, and B + D + F + &c. = ·03071, alſo $\frac{4b}{\sqrt[3]{c^2}}$ being $= \frac{36}{\sqrt[3]{250}}$, we ſhall have

By

By the 1st series				By the latter series			
	·34051	-	log. $\bar{1}$·5321299		·03071	-	log. $\bar{2}$·4872798
	$\frac{36}{\sqrt[3]{350}}$	- -	0·7082798		$\frac{36}{\sqrt[3]{350}}$	- - -	0·7082798
series	= − 1·739441	-	0·2404097	series	= + ·1568771	-	$\bar{1}$·1955596
X	= + 0·948167			X	= − ·9481669		
	− ·791274		the least root		− ·7912898		the same root

Which nearly agree with the same root found in Art. 113.

136. But in Art. 62 the same root was found to be $\frac{3-\sqrt{21}}{2}$, hence then we shall have these first two following equations, and by means of their sum and difference we obtain the other two:

$$\frac{\sqrt[3]{20\sqrt{7}+36}-\sqrt[3]{20\sqrt{7}-36}+\sqrt{21}-3}{72}\sqrt[3]{350}=\frac{1}{3}+\frac{2.5.8.11.81^2}{3.6.9.12.15.175^2}\ \&c.$$

$$\frac{\sqrt[3]{20\sqrt{7}+36}-\sqrt[3]{20\sqrt{7}-36}-\sqrt{21}+3}{72}\sqrt[3]{350}=\frac{2.5.81}{3.6.9.175}+\&c.$$

$$\frac{\sqrt[3]{20\sqrt{7}+36}-\sqrt[3]{20\sqrt{7}-36}}{36}\sqrt[3]{350}=\frac{1}{3}+\frac{2.5.81}{3.6.9.175}+\&c.$$

$$\frac{\sqrt{21}-3}{36}\sqrt[3]{350}=\frac{1}{3}-\frac{2.5.81}{3.6.9.175}+\frac{2.5.8.11.81^2}{3.6.9.12.15.175^2}-\&c.$$

And the last of these agrees with one found in Art. 115.

137. Ex. 4. In the equation $x^3-15x=22$, we have $2b=22$, and $\sqrt[3]{b^2+c^2}=5$; consequently $b=11$, and $c^2=5^3-b^2=125-121=4$, which being less than b^2 or 121, this belongs to the first class of series, or that for the greatest root.

Now

Now $x = \sqrt[3]{b+c} + \sqrt[3]{b-c} = \sqrt[3]{13} + \sqrt[3]{9} = 4{\cdot}4314186 =$ the root of the equation $x^3 - 3\sqrt[3]{117}\,.\,x = 22$. And the terms of the two ſeries being found as in Art. 117, we have the firſt $= A - C - \&c. = 1 - {\cdot}0000450 = {\cdot}9999550$, and the ſecond $= B + D + \&c. = {\cdot}0036731 + {\cdot}0000008 = {\cdot}0036739$. Alſo $4\sqrt[3]{b} = 4\sqrt[3]{11} = \sqrt[3]{704}$. Hence,

By the 1ſt ſeries

	·9999550	- log.	$\bar{1}$·9999805
	$\sqrt[3]{704}$	- - -	0·9491909
ſeries =	+ 8 8955200	-	0·9491714
X =	− 4·4314186		
	+ 4·4641014	greateſt root	

By the 2d ſeries

	·0036739	- log.	$\bar{3}$·5651273
	$\sqrt[3]{704}$	- - -	0·9491909
ſeries =	+ ·0326827		$\bar{2}$·5143182
X =	+ 4·4314186		
	+ 4·4641013	the ſame root	

Which nearly agree with the ſame root found in Art. 117.

138. But in Art. 64, the ſame root was found to be $1 + \sqrt{12} = 1 + 2\sqrt{3}$, hence we obtain theſe two firſt equations following, and their ſum and difference give the other two:

$$\frac{1+\sqrt{12}+\sqrt[3]{13}+\sqrt[3]{9}}{4\sqrt[3]{11}} = 1 - \frac{2.5.8.2^4}{3.6.9.12.11^4} - \frac{2.5.8.11.14.17.20.2^8}{3.6.9.12.15.18.21.24.11^8} - \&c.$$

$$\frac{1+\sqrt{12}-\sqrt[3]{13}-\sqrt[3]{9}}{4\sqrt[3]{11}} = \frac{2.2^2}{3.6.11^2} + \frac{2.5.8.11.14.2^6}{3.6.9.12.15.18.11^6} + \&c.$$

$$\frac{\sqrt[3]{13}+\sqrt[3]{9}}{2\sqrt[3]{11}} = 1 - \frac{2.2^2}{3.6.11^2} - \frac{2.5.8.2^4}{3.6.9.12.11^4} - \frac{2.5.8.11.14.2^6}{3.6.9.12.15.18.11^6} - - \&c.$$

$$\frac{1+\sqrt{12}}{2\sqrt[3]{11}} = 1 + \frac{2.2^2}{3.6.11^2} - \frac{2.5.8.2^4}{3.6.9.12.11^4} + \frac{2.5.8.11.14.2^6}{3.6.9.12.15.18.11^6} - + \&c.$$

The laſt of which agrees with one found in Art. 119.

And thus we may find the ſums of as many ſeries of theſe kinds as we pleaſe; as well as the ſum of any of the general ſeries, by means of the roots of given cubic equations. As to the ſummation of other forms of ſeries by means of the roots of equations of other orders, I ſhall perhaps treat of them on ſome future occaſion.